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The effective stiffness and stress concentrations of a multi-layer laminate

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ABSTRACT

Using the jump conditions for the geometrically linearized strains and the Cauchy stresses and assuming homogeneous strain and stress fields inside the layers, we determine the effective stiffness tetrad of a laminate consisting of linearly elastic layers. From a two-layer reference solution, an explicit solution for the multi-layer case is derived. Unlike classical methods, the present approach applies to any loading case and an arbitrary number of layers with arbitrary stiffness tetrads, and can be considered as the explicit analytical solution of the multi-layer homogenization in linear elasticity.

1. Introduction

Laminates are widely used in industrial applications and aeronautics (see, e.g., [18]), because their properties can be controlled by choosing different material combinations, volume fractions, fiber orientations, etc. Thus, one can design laminates adjusted to ones needs, e.g., a high mass-specific strength in aeronautics, low cost plates in mass productions, and fiber reinforcement that is optimized for specific loading cases.

In order to accelerate simulations for composite materials, a reliable prediction of its effective elastic properties is required. Also, for the design of laminates with specific elastic properties, analytical estimates are needed. The optimization problems that arise in this context are still subject to performance improvement (e.g. [11]). A recent comparison of different analytical models for estimating the effective stiffness due to a wavy fiber alignment can be found in Nik et al. [17]. A comparison of an analytical, semi-analytical and numerical estimates can be found in Rodríguez-Ramos [20].

In the present paper, we present the analytical effective stiffness of a multi-layer laminate with arbitrary anisotropic stiffness tetrads and volume fractions. The concentration tensors for calculating the partial stresses and strains from the effective stresses and strains and are obtained as well. The partial stresses are usually needed for predicting failure of the laminate (e.g. [12]). The result is obtained by considering the *jump balances of stresses and strains* and presuming *piecewise homogeneous stress and strain fields* in the individual layers. A similar proceeding has been employed by Martínez et al. [16], who considered the isostrain condition parallel and the isostress condition perpendicular to the interface to estimate the delamination resistance. Likewise, the jump conditions have been used by Idiart [10] to give expressions for the effective stress and strain potentials of infinite-rank laminates of two phases, where the laminates at the different scales have different orientations. Here we are concerned with arbitrary many layers, but only one interface orientation. However, the rank-1 case in Idiart [10] corresponds to the double-layer laminate in this work.

The engineers approach is to consider specific loading cases, like tension/compression parallel and perpendicular to the laminate normal, shearing parallel and perpendicular to the laminate, with specific material symmetries and a symmetric alignment of the anisotropy axes. For each of theses loading cases, specific rules of mixture apply to specific elastic constants. For example, in a tension test parallel to an interface of a laminate of isotropic constituents, the effective Young's modulus is given by the Voigt average, i.e. the arithmetic mean of the Young's moduli. However, this approach works only for special layer arrangements [5], for which the notion of the effective laminate behavior is obvious.

The analytical effective stiffness of a double layer composite with arbitrary layer stiffnesses has been given already by Francfort and Murat [6] (see Section 4.1). This work is, however, mainly concerned with bounds, and extremely mathematical, so it is hard to extract explicit results for engineering applications.[1] The present

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[1] For example, the restriction to the three-dimensional space comes after the analytical homogenization of the double layer laminate.

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article lays out a more conventional proceeding, with nevertheless quite general but also practical results. We find explicit symbolic expressions for the effective stiffness of a m-layer laminate. The explicit expression for a two-layer composite can already be extracted from Francfort and Murat [6], while the explicit expression for the multi-layer case appears to be new.

1.1. Jump balances

In the context of linear elasticity, the jump balances have been exploited by, e.g., Laws [13], Gemperlová et al. [7], and Dvorak [5]. The principle result is that the jump of the stresses and strains can be calculated using only the stiffness tetrads on both sides and the stresses and strains on one side of the interface. However, the focus is on stress concentration factors rather than effective stiffness tetrads. The jump balances have firstly been used by Francfort and Murat [6] to determine the effective laminate stiffness, but the method remained somewhat unnoticed.

1.2. Piecewise homogeneous stress and strain fields

The assumption of homogeneous stresses and strains inside the lamellae is quite common [2]. Here, it is tacitly implied by the rule of mixture for the stresses and strains. The assumption is particularly reasonable for thin layers: fluctuations are energetically unfavorable, i.e., in the absence of microscopic perturbations (other than the interfaces), the stress and strain fields are homogeneous in each layer. If the lamellae thickness is similar to that of the whole laminate, the homogeneity is rendered unrealistic by bending moments. This corresponds to the well-known loss of scale separation between the microscale and the macroscale [8]. In this case, the layers need to be considered explicitly in the engineering problem. Particular methods have been developed to this end, see, e.g., Makeev and Armanios [15].

1.3. Outline

First, we give the general solution for the effective stiffness tetrad, the stress concentrations, and the strain concentrations in the two-layer case. Then, we investigate two examples in more detail: (a) two isotropic layers and (b) two transversely isotropic layers of equal stiffness and volume fraction but different fiber direction. Next, we consider the multi-layer case. Based on the two-layer-solution, we derive an explicit expression for the effective stiffness tetrad. Finally, we compare our result to other approaches.

1.4. Notation

A direct notation is preferred. Vectors are denoted as bold minuscules (like $\boldsymbol{a}$), second-order tensors as bold majuscules or bold greek letters (like $\boldsymbol{A}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}$), and fourth-order tensors as super bold majuscules (like $\mathbb{A}$). The dyadic product and single scalar contractions are denoted like $(\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{c})\cdot\cdot(\boldsymbol{d}\otimes\boldsymbol{e}) = (\boldsymbol{b}\cdot\boldsymbol{d})(\boldsymbol{c}\cdot\boldsymbol{e})\boldsymbol{a}$, with $\cdot$ being the usual scalar product between vectors. The upper index T denotes the transpose of a second-order tensor, $(\boldsymbol{a}\otimes\boldsymbol{b})^T := \boldsymbol{b}\otimes\boldsymbol{a}$. The upper index S is used to define symmetric parts for second- and fourth-order tensors,

$$\boldsymbol{A}^S := \frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}^T), \tag{1}$$

$$\mathbb{A}^S := \frac{1}{4}(A_{ijkl}+A_{jikl}+A_{ijlk}+A_{jilk})\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}_l. \tag{2}$$

Components are given w.r.t. orthonormalized bases $\{\boldsymbol{e}_i\}$ or $\{\boldsymbol{E}_I\}$, where

$$\begin{aligned}
\boldsymbol{E}_1 &:= \boldsymbol{e}_1\otimes\boldsymbol{e}_1, & \boldsymbol{E}_4 &:= \sqrt{2}(\boldsymbol{e}_2\otimes\boldsymbol{e}_3)^S,\\
\boldsymbol{E}_2 &:= \boldsymbol{e}_2\otimes\boldsymbol{e}_2, & \boldsymbol{E}_5 &:= \sqrt{2}(\boldsymbol{e}_3\otimes\boldsymbol{e}_1)^S,\\
\boldsymbol{E}_3 &:= \boldsymbol{e}_3\otimes\boldsymbol{e}_3, & \boldsymbol{E}_6 &:= \sqrt{2}(\boldsymbol{e}_1\otimes\boldsymbol{e}_2)^S.
\end{aligned}$$

We make use of Einstein's summation convention with implicit summation from 1 to 3 and 1 to 6 for minuscule and majuscule indices that appear pairwise in a product, e.g., the second order identity tensor is given by

$$\boldsymbol{I} = \boldsymbol{e}_i\otimes\boldsymbol{e}_i = \boldsymbol{E}_1+\boldsymbol{E}_2+\boldsymbol{E}_3. \tag{3}$$

Within this paper, all relevant fourth-order tensors possess the left and right subsymmetry, i.e., $A_{ijkl} = A_{jikl} = A_{ijlk} = A_{jilk}$. The inverse of such fourth-order tensors satisfies $\mathbb{A}\cdot\cdot\mathbb{A}^{-1} = \mathbb{I}^S$, where

$$\mathbb{I}^S = \frac{1}{2}(\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_i\otimes\boldsymbol{e}_j + \boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_i) = \boldsymbol{E}_I\otimes\boldsymbol{E}_I \tag{4}$$

denotes the fourth order identity tensor on symmetric second-order tensors.

2. Solution for a two-layer laminate

For a two-layer laminate, we have to consider only one interface with normal $\boldsymbol{n}$. The quantities on different sides of the interface are distinguished by upper indices + and −. The jump balances for the Cauchy stresses ($\boldsymbol{\sigma}$) and the strains ($\boldsymbol{\varepsilon}$) and the volume average of these quantities are given by

$$\boldsymbol{\varepsilon}^+ - \boldsymbol{\varepsilon}^- = (\boldsymbol{a}\otimes\boldsymbol{n})^S, \tag{5}$$

$$(\boldsymbol{\sigma}^+ - \boldsymbol{\sigma}^-)\cdot\boldsymbol{n} = \boldsymbol{o}, \tag{6}$$

$$V := V^+ + V^-, \tag{7}$$

$$V^\pm \geqslant 0, \tag{8}$$

$$\boldsymbol{\varepsilon} := \frac{1}{V}(V^+\boldsymbol{\varepsilon}^+ + V^-\boldsymbol{\varepsilon}^-), \tag{9}$$

$$\boldsymbol{\sigma} := \frac{1}{V}(V^+\boldsymbol{\sigma}^+ + V^-\boldsymbol{\sigma}^-), \tag{10}$$

where $\boldsymbol{a}\otimes\boldsymbol{n}$ is the jump of the deformation gradient at the interface. Inside the layers we assume Hooke's law,

$$\boldsymbol{\sigma}^\pm = \mathbb{C}^\pm\cdot\cdot\boldsymbol{\varepsilon}^\pm, \tag{11}$$

with the stiffness tetrads $\mathbb{C}^\pm$. Subsequently, we will use abbreviations to denote volume fractions, the difference of stiffness tetrads and the Voigt average

$$v^\pm := \frac{V^\pm}{V}, \tag{12}$$

$$\Delta\mathbb{C} := \mathbb{C}^+ - \mathbb{C}^-, \tag{13}$$

$$\mathbb{C}^V := v^+\mathbb{C}^+ + v^-\mathbb{C}^-. \tag{14}$$

Next, we seek an explicit expression for $\mathbb{C}$ in $\boldsymbol{\sigma} = \mathbb{C}\cdot\cdot\boldsymbol{\varepsilon}$. Inserting the constitutive equations (Eq. 11) into the jump balance of the stresses (Eq. 6) gives

$$(\mathbb{C}^+\cdot\cdot\boldsymbol{\varepsilon}^+ - \mathbb{C}^-\cdot\cdot\boldsymbol{\varepsilon}^-)\cdot\boldsymbol{n} = \boldsymbol{o}. \tag{15}$$

Using the jump balance for the strains (Eq. 5), we can eliminate either $\boldsymbol{\varepsilon}^+$ or $\boldsymbol{\varepsilon}^-$, thus

$$(\mathbb{C}^+\cdot\cdot(\boldsymbol{a}\otimes\boldsymbol{n})^S + \Delta\mathbb{C}\cdot\cdot\boldsymbol{\varepsilon}^-)\cdot\boldsymbol{n} = \boldsymbol{o}, \tag{16}$$

$$(\mathbb{C}^-\cdot\cdot(\boldsymbol{a}\otimes\boldsymbol{n})^S + \Delta\mathbb{C}\cdot\cdot\boldsymbol{\varepsilon}^+)\cdot\boldsymbol{n} = \boldsymbol{o}. \tag{17}$$

Since the stiffness tetrads possess the sub-symmetries, we can drop the symmetrization of $\boldsymbol{a}\otimes\boldsymbol{n}$ and rewrite these equations

$$\boldsymbol{A}^+\cdot\boldsymbol{a} = -(\Delta\mathbb{C}\cdot\cdot\boldsymbol{\varepsilon}^-)\cdot\boldsymbol{n}, \tag{18}$$

$$\boldsymbol{A}^-\cdot\boldsymbol{a} = -(\Delta\mathbb{C}\cdot\cdot\boldsymbol{\varepsilon}^+)\cdot\boldsymbol{n}. \tag{19}$$

The so-called acoustic tensors $\boldsymbol{A}^{\pm}$ are associated to the stiffness tetrads where $A_{ik} := C_{ijkl} n_j n_l$ are the components w.r.t. an orthonormal basis. Multiplying the latter equations with the volume fractions v^- and v^+, respectively, their sum becomes

$$(v^-\boldsymbol{A}^+ + v^+\boldsymbol{A}^-)\cdot\boldsymbol{a} = -(\Delta\mathbb{C}\cdot\cdot\,\varepsilon)\cdot\boldsymbol{n}, \tag{20}$$

where the rule of mixture for the strains (Eq. 9) has been used. This equation can be used to determine $\boldsymbol{a}$. For convenience, we define the tensors $\boldsymbol{Z}$ and $\mathbb{Z}$ via

$$\boldsymbol{Z} := (v^-\boldsymbol{A}^+ + v^+\boldsymbol{A}^-)^{-1}, \tag{21}$$
$$\mathbb{Z} := \boldsymbol{n}\otimes\boldsymbol{Z}\otimes\boldsymbol{n}, \tag{22}$$

which renders a compact expression for $\boldsymbol{n}\otimes\boldsymbol{a}$

$$\boldsymbol{n}\otimes\boldsymbol{a} = -\mathbb{Z}\cdot\cdot\,\Delta\mathbb{C}\cdot\cdot\,\varepsilon. \tag{23}$$

With the stiffness tetrads being positive definite, the acoustic tensors are positive definite, too, and so is their weighted sum. Thus $\boldsymbol{Z}$ exists and is positive definite. The intermediate fourth-order tensor $\mathbb{Z}$ possesses neither the major symmetry nor one of the subsymmetries. With the subsymmetries of $\mathbb{C}^{\pm}$ and the symmetrization of $\boldsymbol{n}\otimes\boldsymbol{a}$ we can impose the subsymmetries without loss of generality, thus replacing $\mathbb{Z}$ by $\mathbb{Z}^S$. Then, due to the symmetry of $\boldsymbol{Z}$, $\mathbb{Z}^S$ also possesses the major symmetry. The strain jump is conveniently rewritten as

$$(\boldsymbol{a}\otimes\boldsymbol{n})^S = -\mathbb{Z}^S\cdot\cdot\,\Delta\mathbb{C}\cdot\cdot\,\varepsilon. \tag{24}$$

On the other hand, combining the constitutive equations (Eq. 11) and the rule of mixture for the stresses (Eq. 10), we find the effective stresses,

$$\sigma = v^+\mathbb{C}^+\cdot\cdot\,\varepsilon^+ + v^-\mathbb{C}^-\cdot\cdot\,\varepsilon^-. \tag{25}$$

By means of the jump balance of the strains (Eq. 5), we can remove either ε^- or ε^+,

$$\sigma = (v^+\mathbb{C}^+ + v^-\mathbb{C}^-)\cdot\cdot\,\varepsilon^+ + v^-\mathbb{C}^-\cdot\cdot\,(\boldsymbol{a}\otimes\boldsymbol{n})^S, \tag{26}$$
$$\sigma = (v^+\mathbb{C}^+ + v^-\mathbb{C}^-)\cdot\cdot\,\varepsilon^- - v^+\mathbb{C}^+\cdot\cdot\,(\boldsymbol{a}\otimes\boldsymbol{n})^S. \tag{27}$$

The first term on either right hand side contains the Voigt average. Again, multiplying the latter equations by v^+ and v^-, their sum provides the effective stiffness $\mathbb{C}$

$$\sigma = \mathbb{C}\cdot\cdot\,\varepsilon, \tag{28}$$
$$\mathbb{C} := \mathbb{C}^V - v^+v^-\Delta\mathbb{C}\cdot\cdot\,\mathbb{Z}^S\cdot\cdot\,\Delta\mathbb{C}, \tag{29}$$

where Eqs. (9) and (24) have been used. The result is reasonable: As expected, $\mathbb{C}$ is invariant under a change of the phase indices $+\leftrightarrow-$ and inherits the major symmetry and both subsymmetries from $\mathbb{C}^{\pm}$. It is also invariant under a scaling or a change of sign of $\boldsymbol{n}$. For the extremal cases $\mathbb{C}^+ = \mathbb{C}^-$, $v^+ = 0$ or $v^- = 0$, we obtain the stiffness tetrad of a homogeneous medium. The second summand in Eq. (29) is negative semi-definite, thus the Voigt bound cannot be exceeded. Further plausibility tests are postponed to Section 2.2, where isotropic phases are considered.

2.1. Concentration tensors

Using Eqs. (9), (5), and (24) defining the $\mathbb{K}$- and $\mathbb{L}$-tensors by

$$\mathbb{K}^{\pm} := \mathbb{I}^S \mp v^{\mp}\mathbb{Z}^S\cdot\cdot\,\Delta\mathbb{C}, \tag{30}$$
$$\mathbb{L}^{\pm} := \mathbb{C}^{\pm}\cdot\cdot\,\mathbb{K}^{\pm}\cdot\cdot\,\mathbb{C}^{-1}, \tag{31}$$

we can express the partial strains and stresses in terms of the effective strains and stresses, respectively,

$$\varepsilon^{\pm} = \mathbb{K}^{\pm}\cdot\cdot\,\varepsilon, \tag{32}$$
$$\sigma^{\pm} = \mathbb{L}^{\pm}\cdot\cdot\,\sigma. \tag{33}$$

This concept is due to Hill [9]. The non-dimensional concentration tensors fulfill

$$\mathbb{I}^S = v^+\mathbb{K}^+ + v^-\mathbb{K}^-, \tag{34}$$
$$\mathbb{I}^S = v^+\mathbb{L}^+ + v^-\mathbb{L}^-. \tag{35}$$

In like vein, we can relate the partial stresses and the effective strains via $\mathbb{M}$-tensors

$$\mathbb{M}^{\pm} := \mathbb{C}^{\pm}\cdot\cdot\,\mathbb{K}^{\pm}, \tag{36}$$
$$\sigma^{\pm} = \mathbb{M}^{\pm}\cdot\cdot\,\varepsilon, \tag{37}$$
$$\mathbb{C} = v^+\mathbb{M}^+ + v^-\mathbb{M}^-, \tag{38}$$

as has been done first by Laws [13] (Eqs. (28) and (29) there). All concentration tensors exhibit both subsymmetries but not the major symmetry, in general. The partial stresses are required for estimating the strength of a laminate. With the partial stresses at hand, one can evaluate a flow or failure criterion inside the layers, or calculate the normal traction and shear stresses at the interface. The latter is needed to predict the delamination resistance, see, e.g., Zou et al. [21] and Diaz Diaz and Caron [4].

2.2. Example 1: isotropic layers

Let us consider two isotropic stiffness tetrads, given in terms of the fourth-order isotropic projectors $\mathbb{P}_1$ and $\mathbb{P}_2$,

$$\mathbb{C}^{\pm} = 3K^{\pm}\mathbb{P}_1 + 2G^{\pm}\mathbb{P}_2, \tag{39}$$
$$\mathbb{P}_1 := \frac{1}{3}\boldsymbol{I}\otimes\boldsymbol{I}, \tag{40}$$
$$\mathbb{P}_2 := \mathbb{I}^S - \frac{1}{3}\boldsymbol{I}\otimes\boldsymbol{I}. \tag{41}$$

For the subsequent sections, the following abbreviations come in handy,

$$A^{\pm} := C^{\pm}_{11} = K^{\pm} + \frac{4}{3}G^{\pm}, \quad 3K^{\pm} = A^{\pm} + 2B^{\pm}, \tag{42}$$
$$B^{\pm} := C^{\pm}_{12} = K^{\pm} - \frac{2}{3}G^{\pm}, \quad 2G^{\pm} = A^{\pm} - B^{\pm}. \tag{43}$$

A and B correspond to $k+m$ and $k-m$, with the moduli k and m as used by Hill [9].

Furthermore, the following combinations (difference, Voigt average, mixed average, Reuss average) will appear frequently in subsequent sections,

$$\Delta X := X^+ - X^-, \tag{44}$$
$$X^V := v^+X^+ + v^-X^-, \tag{45}$$
$$X^M := v^+X^- + v^-X^+, \tag{46}$$
$$X^R := \left(v^+(X^+)^{-1} + v^-(X^-)^{-1}\right)^{-1}, \tag{47}$$
$$= X^V - v^+v^-\frac{\Delta X^2}{X^M}, \tag{48}$$
$$= \frac{X^+X^-}{X^M}, \tag{49}$$

where $X \in \{A, B, G, K, \boldsymbol{A}, \mathbb{C}\}$. W.r.t. the basis $\{\boldsymbol{E}_I\otimes\boldsymbol{E}_J\}$, the components are

$$\mathbb{C}^{\pm} = \begin{bmatrix} C^{\pm}_{11} & C^{\pm}_{12} & C^{\pm}_{12} & & & \\ C^{\pm}_{12} & C^{\pm}_{11} & C^{\pm}_{12} & & & \\ C^{\pm}_{12} & C^{\pm}_{12} & C^{\pm}_{11} & & & \\ & & & C^{\pm}_{11}-C^{\pm}_{12} & & \\ & & & & C^{\pm}_{11}-C^{\pm}_{12} & \\ & & & & & C^{\pm}_{11}-C^{\pm}_{12} \end{bmatrix}. \tag{50}$$

The effective stiffness tetrad becomes transversely isotropic w.r.t. the interface normal $\boldsymbol{n}$. For $\boldsymbol{n} = \boldsymbol{e}_3$, the non-zero components of $\mathbb{C}$ w.r.t. $\{\boldsymbol{E}_I \otimes \boldsymbol{E}_J\}$ are given by

$$\mathbb{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{11} & C_{13} & & & \\ C_{13} & C_{13} & C_{33} & & & \\ & & & C_{44} & & \\ & & & & C_{44} & \\ & & & & & C_{11}-C_{12} \end{bmatrix}, \tag{51}$$

where

$$C_{11} = A^V - v^- v^+ \frac{\Delta B^2}{A^M}, \tag{52}$$

$$C_{12} = B^V - v^- v^+ \frac{\Delta B^2}{A^M}, \tag{53}$$

$$C_{13} = B^V - v^- v^+ \frac{\Delta A \Delta B}{A^M}, \tag{54}$$

$$C_{33} = A^R, \tag{55}$$

$$C_{44} = 2G^R, \tag{56}$$

$$C_{66} = C_{11} - C_{12} = 2G^V. \tag{57}$$

$2G^V$ and $2G^R$ are double eigenvalues of $\mathbb{C}$. The corresponding eigendeformations $\varepsilon^{V/R}$ are given by arbitrary linear combinations (with coefficients $l_i^{V/R}$)

$$\varepsilon^V = l_1^V \boldsymbol{E}_6 + l_2^V \frac{1}{\sqrt{2}}(\boldsymbol{E}_1 - \boldsymbol{E}_2), \tag{58}$$

$$\varepsilon^R = l_1^R \boldsymbol{E}_4 + l_2^R \boldsymbol{E}_5. \tag{59}$$

Thus, depending on whether a shear deformation $\boldsymbol{F} = \boldsymbol{I} + \gamma \boldsymbol{d} \otimes \boldsymbol{s}$ has either $\boldsymbol{d}$ or $\boldsymbol{s}$ parallel and the other one perpendicular to the interface (Reuss case) or both $\boldsymbol{d}$ and $\boldsymbol{s}$ parallel to the interface (Voigt case), one obtains the correct shear modulus, i.e., the lamellae appear in sequence or parallel w.r.t. the shear deformation. These eigendeformations are independent on $K^\pm$ and $G^\pm$.

The remaining two eigendeformations depend on $K^\pm$ and $G^\pm$,

$$\varepsilon_5^\alpha = \frac{\cos\alpha}{\sqrt{2}}(\boldsymbol{E}_1 + \boldsymbol{E}_2) + \sin\alpha \boldsymbol{E}_3, \tag{60}$$

$$\varepsilon_6^\alpha = -\frac{\sin\alpha}{\sqrt{2}}(\boldsymbol{E}_1 + \boldsymbol{E}_2) + \cos\alpha \boldsymbol{E}_3, \tag{61}$$

where the parameter α is determined by

$$\cot 2\alpha = \frac{(v^+)^2 B^+ A^- + (v^-)^2 B^- A^+ + v^+ v^- (\Delta A^2 - 2\Delta B^2 + A^+ B^+ + A^- B^-)}{2\sqrt{2}(v^+ B^+ A^- + v^- B^- A^+)}. \tag{62}$$

In the event of $G^+ = G^-$, the laminate stiffness becomes isotropic. This is an example for the fundamental result of Hill [9], who found that any microstructure of two isotropic materials with equal shear rigidity is isotropic, irrespective of the geometry of the microstructure.

For the concentration tensors $\mathbb{K}^+$ and $\mathbb{L}^+$, we find the following components w.r.t. $\boldsymbol{E}_I \otimes \boldsymbol{E}_J$,

$$\mathbb{K}^\pm = \begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ K_{31}^\pm & K_{31}^\pm & K_{33}^\pm & & & \\ & & & K_{44}^\pm & & \\ & & & & K_{44}^\pm & \\ & & & & & 1 \end{bmatrix}, \tag{63}$$

$$\mathbb{L}^\pm = \begin{bmatrix} L_{11}^\pm & L_{12}^\pm & L_{13}^\pm & & & \\ L_{12}^\pm & L_{11}^\pm & L_{13}^\pm & & & \\ 0 & 0 & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & L_{11}^\pm - L_{12}^\pm \end{bmatrix}, \tag{64}$$

where

$$K_{31}^+ = -v^- \frac{\Delta B}{A^M}, \quad L_{11}^+ = \frac{G^+}{G^V}\left(1 + \frac{v^- G^- (A^- B^+ - A^+ B^-)}{3(v^- G^- K^- A^+ + v^+ G^+ K^+ A^-)}\right), \tag{65}$$

$$K_{33}^+ = \frac{A^-}{A^M}, \quad L_{12}^+ = \frac{G^+}{G^V} \frac{v^- G^- (A^- B^+ - A^+ B^-)}{3(v^- G^- K^- A^+ + v^+ G^+ K^+ A^-)}, \tag{66}$$

$$K_{44}^+ = \frac{G^-}{G^M}, \quad L_{13}^+ = -v^- \frac{(A^+)^2 B^- - (A^-)^2 B^+ + B^+ B^- (\Delta A - 2\Delta B)}{6(v^- G^- K^- A^+ + v^+ G^+ K^+ A^-)}, \tag{67}$$

and $$L_{66}^+ = L_{11}^+ - L_{12}^+ = \frac{G^+}{G^V}. \tag{68}$$

The counterparts $\mathbb{K}^-$ and $\mathbb{L}^-$ are easily determined by virtue of Eqs. (34) and (35), respectively.

2.2.1. Comparison to other results

2.2.1.1. Classical laminate theory. In the classical laminate theory (CLT), a plane stress state is usually presumed, i.e. there are only three non-zero in-plane stress components (σ_{11}, σ_{22} and σ_{12} when $\boldsymbol{e}_3$ is the laminate normal), coupled by a Q_{ij}-matrix to the three independent strain components $\varepsilon_{11}, \varepsilon_{22}$ and ε_{12} [3]. Thus, Hooke's matrix is reduced to an effective 3×3 matrix Q_{ij} for a plane stress state, which is obtained by adding the individual plane-stress Hooke matrices $Q_{ij}^\pm$ of the individual materials, weighted by the layer thickness fraction. Considering this special case, we found full agreement with the CLT approach by applying a plane stress state to the inverse of $\mathbb{C}$ (given in Section 2.2, Eq. 51), inverting the relations of the independent strain components and extracting the reduced plane strain stiffness.

2.2.1.2. Liu et al. [14]. The work of Liu et al. [14] follows the proceeding outlined in the introduction: for isotropic phases, the effective transversally isotropic laminate stiffness is presumed, and for some characteristic in-plane and normal-to-plane tests the corresponding kinematic and dynamic equivalences are exploited to derive the compliance constants. We found all given effective compliances (Eqs. (21), (22), (29), (30), (35), and (36) in Liu et al. [14]) in agreement with our results.

2.3. Example 2: two transversely isotropic layers, orthotropic laminate

Another important case is the stacking of unidirectionally reinforced layers. These layers exhibit a transversely isotropic stiffness tetrad with the fiber direction and symmetry axis $\boldsymbol{d}$,

$$\begin{aligned} \mathbb{C}^t(\boldsymbol{d}) := {} & C_{12}^t \boldsymbol{I} \otimes \boldsymbol{I} + (C_{11}^t - C_{12}^t)\mathbb{I}^S \\ & + 2(C_{12}^t - C_{11}^t + C_{44}^t)(\boldsymbol{d} \otimes \boldsymbol{I} \otimes \boldsymbol{d})^S && (69) \\ & + (C_{13}^t - C_{12}^t)(\boldsymbol{I} \otimes \boldsymbol{d} \otimes \boldsymbol{d} + \boldsymbol{d} \otimes \boldsymbol{d} \otimes \boldsymbol{I}) \\ & + (C_{11}^t + C_{33}^t - 2C_{13}^t - 2\,C_{44}^t)\boldsymbol{d} \otimes \boldsymbol{d} \otimes \boldsymbol{d} \otimes \boldsymbol{d}. && (70) \end{aligned}$$

For $\boldsymbol{d} = \boldsymbol{e}_3$, the components w.r.t. the basis $\{\boldsymbol{E}_I \otimes \boldsymbol{E}_J\}$ are given by

$$\mathbb{C}^t(\boldsymbol{e}_3) = \begin{bmatrix} C_{11}^t & C_{12}^t & C_{13}^t & & & \\ C_{12}^t & C_{11}^t & C_{13}^t & & & \\ C_{13}^t & C_{13}^t & C_{33}^t & & & \\ & & & C_{44}^t & & \\ & & & & C_{44}^t & \\ & & & & & C_{11}^t - C_{12}^t \end{bmatrix} \boldsymbol{E}_I \otimes \boldsymbol{E}_J. \tag{71}$$

The components after a change of the symmetry axis from $\boldsymbol{e}_3$ to $\boldsymbol{e}_1$ are obtained by interchanging the first with the third and the fourth with the sixth rows and columns.

We consider the most common case: both layers consist of the same material but differ by the fiber direction. For both layers the fiber direction is parallel to the interface, i.e., $\boldsymbol{d}^{\pm} \cdot \boldsymbol{n} = 0$. Both layers have a volume fraction of 50%. In order not to conceal the symmetries of the effective stiffness tetrad, we choose an orthonormalized basis $\{\boldsymbol{e}_i\}$ w.r.t. which

$$\boldsymbol{d}^{\pm} = \cos\phi \boldsymbol{e}_1 \pm \sin\phi \boldsymbol{e}_2, \tag{72}$$
$$\boldsymbol{n} = \boldsymbol{e}_3. \tag{73}$$

Thus, the directions enclose an angle of 2ϕ, see Fig. 1. The effective stiffness tetrad is orthotropic w.r.t. the basis $\{\boldsymbol{e}_i\}$, and w.r.t. the basis $\{\boldsymbol{E}_I \otimes \boldsymbol{E}_J\}$, its components are given by

$$\mathbb{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{22} & C_{23} & & & \\ C_{13} & C_{23} & C_{33} & & & \\ & & & C_{44} & & \\ & & & & C_{55} & \\ & & & & & C_{66} \end{bmatrix}, \tag{74}$$

where

$$C_{11} = C_{11}^t \sin^2\phi + C_{33}^t \cos^2\phi - x\cos^2\phi\sin^2\phi, \tag{75}$$
$$C_{22} = C_{11}^t \cos^2\phi + C_{33}^t \sin^2\phi - x\cos^2\phi\sin^2\phi, \tag{76}$$
$$C_{33} = C_{11}^t, \tag{77}$$
$$C_{12} = C_{13}^t + x\cos^2\phi\sin^2\phi, \tag{78}$$
$$C_{13} = C_{12}^t \sin^2\phi + C_{13}^t \cos^2\phi, \tag{79}$$
$$C_{23} = C_{12}^t \cos^2\phi + C_{13}^t \sin^2\phi, \tag{80}$$
$$C_{44} = \frac{(C_{11}^t - C_{12}^t)C_{44}^t}{(C_{11}^t - C_{12}^t)\sin^2\phi - C_{44}^t\cos^2\phi}, \tag{81}$$
$$C_{55} = \frac{(C_{11}^t - C_{12}^t)C_{44}^t}{(C_{11}^t - C_{12}^t)\cos^2\phi - C_{44}^t\sin^2\phi}, \tag{82}$$
$$C_{66} = C_{44}^t + 2\left(x - \frac{(C_{13}^t - C_{12}^t)^2}{C_{11}^t}\right)\cos^2\phi\sin^2\phi, \tag{83}$$
$$x := C_{11}^t + C_{33}^t - 2(C_{13}^t + C_{44}^t). \tag{84}$$

If the angle is an integer multiple of $\frac{\pi}{2}$, the fiber directions coincide (up to the sign), and the original stiffness tetrad is recovered, albeit with a different axis of transversal isotropy.

2.3.1. Comparison to other results

Similar to the proceeding in Section 2.2.1.1, the CLT-estimate is given by averaging the plane-stress stiffnesses of the individual layers. We found complete agreement, independent on the angle ϕ and the volume fractions. The plane-stress elastic law is then given by the matrix–vector product

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{1}{8C_{11}^t} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ & Q_{22} & Q_{23} \\ \text{sym} & & Q_{33} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix}, \tag{85}$$

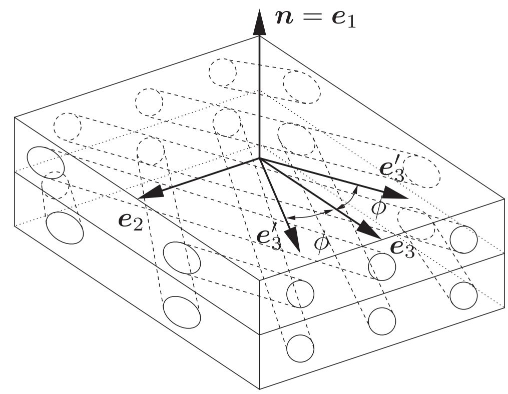

Fig. 1. Two transversely isotropic layers with equal stiffness tetrads and volume fractions.

with

$$\begin{aligned} Q_{11} = {} & 3C_{11}^{t2} - 3C_{12}^{t2} + 2C_{11}^t C_{13}^t - 2C_{12}^t C_{13}^t - 3C_{13}^{t2} + 3C_{11}^t C_{33}^t \\ & + 2C_{11}^t C_{44}^t - 4(C_{11}^{t2} - C_{12}^{t2} + C_{13}^{t2} - C_{11}^t C_{33}^t)\cos 2\phi + (C_{11}^{t2} \\ & - (C_{12}^t - C_{13}^t)^2 + C_{11}^t(C_{33}^t - 2C_{13}^t - 2C_{44}^t))\cos 4\phi, \end{aligned} \tag{86}$$

$$\begin{aligned} Q_{12} = {} & C_{11}^{t2} - C_{12}^{t2} - C_{13}^t(6C_{12}^t + C_{13}^t) + C_{11}^t(6C_{13}^t + C_{33}^t - 2C_{44}^t) \\ & + ((C_{12}^t - C_{13}^t)^2 - C_{11}^{t2} + C_{11}^t(2C_{13}^t - C_{33}^t + 2C_{44}^t)) \\ & \times \cos 4\phi, \end{aligned} \tag{87}$$

$$\begin{aligned} Q_{22} = {} & 3C_{11}^{t2} - 3C_{12}^{t2} + 2C_{11}^t C_{13}^t - 2C_{12}^t C_{13}^t - 3C_{13}^{t2} + 3C_{11}^t C_{33}^t \\ & + 2C_{11}^t C_{44}^t + 4(C_{11}^{t2} - C_{12}^{t2} + C_{13}^{t2} - C_{11}^t C_{33}^t)\cos 2\phi + (C_{11}^{t2} \\ & - (C_{12}^t - C_{13}^t)^2 + C_{11}^t(C_{33}^t - 2C_{13}^t - 2C_{44}^t))\cos 4\phi, \end{aligned} \tag{88}$$

$$\begin{aligned} Q_{33} = {} & 2[C_{11}^{t2} - (C_{12}^t - C_{13}^t)^2 + C_{11}^t(C_{33}^t - 2C_{13}^t + 2C_{44}^t) \\ & + ((C_{12}^t - C_{13}^t)^2 - C_{11}^{t2} + C_{11}^t(2C_{13}^t - C_{33}^t + 2C_{44}^t)) \\ & \times \cos 4\phi], \end{aligned} \tag{89}$$

independent of the volume fractions and

$$\begin{aligned} Q_{13} = {} & 4\Delta v(C_{12}^{t2} - C_{11}^{t2} - C_{13}^{t2} + C_{11}^t C_{33}^t + (C_{11}^{t2} - (C_{12}^t - C_{13}^t)^2 \\ & + C_{11}^t(C_{33}^t - 2C_{13}^t - 2C_{44}^t))\cos 2\phi)\sin 2\phi, \end{aligned} \tag{90}$$

$$\begin{aligned} Q_{23} = {} & 4\Delta v(C_{12}^{t2} - C_{11}^{t2} - C_{13}^{t2} + C_{11}^t C_{33}^t - (C_{11}^{t2} - (C_{12}^t - C_{13}^t)^2 \\ & + C_{11}^t(C_{33}^t - 2C_{13}^t - 2C_{44}^t))\cos 2\phi)\sin 2\phi, \end{aligned} \tag{91}$$

where $v_{\pm}$ are the volume fractions of the layers that are rotated by $\pm\phi$ around the $\boldsymbol{e}_3$-axis.

3. Arbitrary many layers

Subsequently, we encounter only double contractions among fourth and second order tensors. For the sake of simplicity, we drop the respective symbol $\cdot\cdot$ in the remainder.

In the two-layer case, there is only one interface, i.e. one can assign all quantities from the two-layer case to the interface. In order to extend the two-layer solution to the multi-layer solution, we supplement all quantities by an interface index i and can apply the two-layer solution to any two layers that are adjoint to the same interface. For an m-layer composite, the indexing fulfills

$$X_i^+ = X_{i+1}^- \quad \text{modulo} \quad m, \quad X \in \{V, \varepsilon, \sigma, \boldsymbol{A}, \mathbb{C}\}. \tag{92}$$

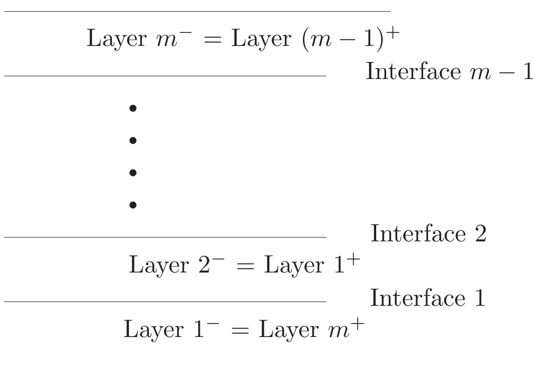

Fig. 2. Indexing of the multi-layer.

For an illustration of the indexing, see Fig. 2. The effective stiffness of the multi-layer laminate is constructed by means of the stiffness tetrads and concentration tensors (which themselves are defined in terms of stiffness tetrads, volume fractions and the interface normal). Using these, we define $\mathbb{A}$- and $\mathbb{B}$-tensors

$$\mathbb{A}_i^- := \mathbb{K}_i^+ (\mathbb{K}_i^-)^{-1}, \tag{93}$$
$$\mathbb{B}_i^- := \mathbb{C}_{i+1}^- \mathbb{A}_i^- \mathbb{C}_i^{--1}, \tag{94}$$

and find the recurrence relations for the partial strains

$$\begin{aligned} \varepsilon_{i+1}^- &= \varepsilon_i^+ && (95)\\ &= \mathbb{K}_i^+ \varepsilon_i && (96)\\ &= \mathbb{K}_i^+ (\mathbb{K}_i^-)^{-1} \varepsilon_i^- && (97)\\ &= \mathbb{A}_i^- \varepsilon_i^-, && (98) \end{aligned}$$

and the partial stresses

$$\begin{aligned} \sigma_{i+1}^- &= \mathbb{C}_{i+1}^- \varepsilon_{i+1}^- && (99)\\ &= \mathbb{C}_{i+1}^- \mathbb{A}_i^- \varepsilon_i^- && (100)\\ &= \mathbb{C}_{i+1}^- \mathbb{A}_i^- (\mathbb{C}_i^-)^{-1} \sigma_i^- && (101)\\ &= \mathbb{B}_i^- \sigma_i^-. && (102) \end{aligned}$$

The inverse of $\mathbb{K}_i^-$ exists in general. Otherwise we would have a decoupling of the strains of adjoint layers, i.e., we would not consider a proper laminate.

With the conventions $\mathbb{A}_0 := \mathbb{I}^S$ and $\mathbb{B}_0 := \mathbb{I}^S$, any partial strain or stress is easily rewritten as a function of only one partial strain or stress (here: ε_1^- or σ_1^-, respectively)

$$\varepsilon_{i+1}^- = \left(\prod_{k=0}^{i} \mathbb{A}_k^- \right) \varepsilon_1^-, \tag{103}$$
$$\sigma_{i+1}^- = \left(\prod_{k=0}^{i} \mathbb{B}_i^- \right) \sigma_1^-. \tag{104}$$

where the product consists of double contractions. Thus, the effective strains and stresses are recast as a function of only ε_1^- and σ_1^-, respectively. The effective strains and stresses are given by

$$V := \sum_{i=1}^{m} V_i^{\pm}, \tag{105}$$
$$\varepsilon := \frac{1}{V} \sum_{i=1}^{m} V_i^{\pm} \varepsilon_i^{\pm}, \tag{106}$$
$$\sigma := \frac{1}{V} \sum_{i=1}^{m} V_i^{\pm} \sigma_i^{\pm}. \tag{107}$$

Considering the −-part of these definitions and using Eqs. (103) and (104), the effective stresses are compactly rewritten as

$$\varepsilon = \frac{1}{V} \mathbb{E}_1^- \varepsilon_1^-, \tag{108}$$
$$\sigma = \frac{1}{V} \mathbb{S}_1^- \sigma_1^-. \tag{109}$$

where

$$\mathbb{E}_1^- := \sum_{k=1}^{m} V_k^- \prod_{n=0}^{k-1} \mathbb{A}_n^-, \tag{110}$$
$$\mathbb{S}_1^- := \sum_{k=1}^{m} V_k^- \prod_{n=0}^{k-1} \mathbb{B}_n^-. \tag{111}$$

Finally, invoking Hooke's law for the −-phase of interface 1, we find

$$\begin{aligned} \sigma &= \frac{1}{V} \mathbb{S}_1^- \sigma_1^- && (112)\\ &= \frac{1}{V} \mathbb{S}_1^- \mathbb{C}_1^- \varepsilon_1^- && (113)\\ &= \mathbb{S}_1^- \mathbb{C}_1^- (\mathbb{E}_1^-)^{-1} \varepsilon, && (114) \end{aligned}$$

which renders the effective stiffness tetrad

$$\mathbb{C} = \mathbb{S}_1^- \mathbb{C}_1^- (\mathbb{E}_1^-)^{-1}. \tag{115}$$

The explicit expression for $\mathbb{C}$ is quite lengthy when all intermediate quantities are inserted. However, it is easily calculated step by step from the stiffness tetrads $\mathbb{C}_{1\ldots m}$, volume fractions $v_{1\ldots m}$ and the interface normal $\boldsymbol{n}$. Again, the Voigt and Reuss averaging of the shear moduli of isotropic layers are recovered for shears parallel or perpendicular to the interfaces.

A Mathematica notebook that performs this task is provided in the articles supplementary material. To obtain $\mathbb{C}$, one could also use the two-layer solution to summarize successively all layers of a multi-layer. One may also feed the linear system for $\boldsymbol{a}_{1\ldots m-1}$ directly to a computer algebra system. However, for an algorithmic implementation, the explicit expression (115) may prove useful.

For the sake of completeness, we also note that the same procedure can be started from any interface other than 1^-, as well as from the plus side of any interface.

It is noteworthy that an interchanging of two layers of a multi-layer does not affect the effective stiffness. This is clear because of the invariance of the effective stiffness of the two-layer solution under interchanging the indices. Interchanging any two layers in the multi-layer case can be regarded as a sequence of interchanges of two adjoint layers.

The closure that comes through considering the periodic arrangement does not affect the solution. Adding the jump $\varepsilon_1 - \varepsilon_m = (\boldsymbol{a}_m \otimes \boldsymbol{n})^S$ allows to sum up all strain jumps from ε_1 to ε_1, which gives $\sum_{i=1}^{m} (\boldsymbol{a}_i \otimes \boldsymbol{n})^S = \boldsymbol{O}$. This implies $\sum_{i=1}^{m} \boldsymbol{a}_n = \boldsymbol{o}$, which allows to remove one of the $\boldsymbol{a}_i$ immediately. In terms of $\mathbb{A}$- and $\mathbb{B}$-tensors, this closure corresponds to

$$\prod_{k=1}^{m} \mathbb{A}_k = \mathbb{I}^S, \tag{116}$$
$$\prod_{k=1}^{m} \mathbb{B}_k = \mathbb{I}^S. \tag{117}$$

4. Summary and outlook

Using the jump balances for the stresses and strains, Hooke's law for each layer and the assumption of homogeneity inside each layer, we derived the effective elastic stiffness of a laminate with m layers. An explicit expression for the elastic stiffness tetrad is provided. For the two-layer case, two typical examples are given, namely isotropic layers of different volume fractions and

transversely isotropic layers of equal volume fractions and stiffness tetrads but different orientations. The result passes different tests of plausibility, most importantly, the compliance with the classical laminate theory is given. It turns out that the classical laminate theory provides identically the plane-stress part of the analytical effective stiffness.

However, the analysis of the jump balances provides the full effective stiffness tetrad in a straightforward manner, using the volume fractions, the individual stiffness tetrads and the interface orientation. No restrictions regarding the stress or strain state, the symmetry class or the alignment of anisotropy axes are necessary. Nevertheless, for the sake of scale separation, the approach is only applicable when the individual layers are much thinner than the overall laminate.

The given methodology allows to easily extract all partial stresses for a given effective strain or stress. This is an important information for an estimation of the laminate strength. It may also serve for the modeling of the effective plastic behavior of laminates, since one can check the flow condition layer-wise. Further, since the full effective stiffness is given, this analytical homogenization result may serve for higher order estimates by sequential homogenization [1]. One could regard the effective stiffness of a double layer laminate as an interface stiffness, and estimate effective properties of a heterogeneous two-phase material by an interface orientation average, in the sense of Richeton and Berbenni [19]. A somewhat similar proceeding has been presented by Idiart [10], who superimposed laminates with different orientations at different scales to obtain an explicit expression for the effective strain potential of a two-phase composite, where also an interface orientation distribution is used.

Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.compstruct.2014.01.027.

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