DIAGONALIZATION: SYMMETRIC AND HERMITIAN MATRICES

Symmetric and hermitian matrices, which arise in many applications, enjoy the property of always being diagonalizable. Also the set of eigenvectors of such matrices can always be chosen as orthonormal. The diagonalization procedure is essentially the same as outlined in Sec. 5.3, as we will see in our examples.

Example 1  The horizontal motion of the system of masses and springs where all the masses are the same and the springs are the same, can be analyzed by diagonalizing the symmetric matrix

\[ A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \]

Diagonalize \( A \).

Solution We have

\[ \det(A - \lambda I) = \det \begin{pmatrix} 2 - \lambda & -1 \\ -1 & 2 - \lambda \end{pmatrix} = (2 - \lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 \]

so that the eigenvalues are \( \lambda_1 = 3 \) and \( \lambda_2 = 1 \). Eigenvectors are found by solving \((A - \lambda I)X = 0\); these equations and solutions are

\[ \lambda_1 = 3: \quad -1x_1 - 1x_2 = 0 \quad \Rightarrow \quad X_1 = \begin{pmatrix} k \\ -k \end{pmatrix} \]
Now \( X_1 \) and \( X_2 \) are orthogonal since \( X_1 \cdot X_2 = k_j - k_j = 0 \). If we normalize \( X_1 \) and \( X_2 \), we have the choices for eigenvectors

\[
\mathcal{O}_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \quad \mathcal{O}_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}
\]

and \( S = \{ \mathcal{O}_1, \mathcal{O}_2 \} \) forms an orthonormal set. Finally, with

\[
P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}
\]

we can write

\[
\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}
\]

Since \( P \) is an orthogonal matrix, \( P^{-1} = P^T \). Also \( P \) represents a rotation of \( \pi/4 \) radians clockwise.

**Example 2** Diagonalize

\[
A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

Choose \( P \) as an orthogonal matrix.

**Solution** The characteristic equation is \((1 - \lambda)(\lambda^2 - 1) = 0\), which has solutions 1 and \(-1\). The eigenvalue 1 has multiplicity 2, and the eigenvalue \(-1\) is simple (multiplicity 1). Now to determine the eigenvectors, we solve equations.

\[
\lambda_1 = 1: \begin{cases} -x_1 + x_2 + 0x_3 = 0 \\ x_1 - x_2 + 0x_3 = 0 \\ 0x_1 + 0x_2 + 0x_3 = 0 \end{cases} \Rightarrow X_1 = \begin{pmatrix} j \\ j \\ k \end{pmatrix}
\]
We see that corresponding to $\lambda_1 = 1$ the eigenspace is two-dimensional. Therefore, we can choose a basis for $E_{1}$ (the basis problem has returned!) of orthonormal vectors. In fact, choosing first $j = 1, k = 0$ and second $k = 1, j = 0$, we have 

$$V_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

which are orthogonal. Normalizing $V_1$ ($V_2$ is already normalized) yields 

$$O_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \quad \text{and} \quad O_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Not as much work is required for $\lambda_2 = -1$ since the eigenspace for this eigenvalue is one-dimensional. Normalizing $X_2$, we have 

$$O_3 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$$

Therefore 

$$P = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad P^{-1} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix} = P^T$$

Diagonalize $A$: 

The last two examples illustrate the basic results for diagonalization of symmetric matrices.

**Theorem 5.4.1** If $A_{n \times n}$ is symmetric with real entries, then

(a) The eigenvalues are real.

(b) Eigenvectors corresponding to distinct eigenvalues are orthogonal.

(c) The eigenspaces of each eigenvalue have orthogonal bases. The dimension of an eigenspace corresponds to the multiplicity of the eigenvalue.

(d) Matrix $A$ is orthogonally diagonalizable; that is, there exists an orthogonal matrix $P$ such that

$$A = PDP^T$$

(and so $D = P^TAP$)

**Proof.** We prove only parts (a) and (b). Parts (c) and (d) are proved in more advanced texts.

(a) Suppose that $\lambda = a + bi$ and that $X + iY$ is the corresponding eigenvector. Therefore,

$$[A - (a + bi)I](X + iY) = 0 + 0i$$

Carrying out the multiplications and setting real and imaginary parts equal, we find

$$aIX - AX - bIY = 0$$
$$aIY - AY + bIX = 0$$

Now take the dot product of the first equation with $Y$ and the second with $X$ to find

$$aY \cdot IX - Y \cdot AX - bY \cdot IY = Y \cdot 0 = 0$$
$$aX \cdot IY - X \cdot AY + bY \cdot IX = X \cdot 0 = 0$$
or
\[
\begin{align*}
ay^TIX - y^TAX - by^TIX &= 0 \\
ax^TIX - x^TAY + bx^TIX &= 0
\end{align*}
\]

Because $A$ and $I$ are symmetric $I = I^T$ and $A = A^T$. Thus
\[
\begin{align*}
ay^TIX^T - y^TA^TX^T - by^TIX &= 0 \\
ax^TIX - x^TAY + bx^TIX &= 0
\end{align*}
\]

and since the transpose of a product is the product of the transposes in reverse order, we have
\[
\begin{align*}
a(x^TIX)^T - (x^TAY)^T - by^TIX &= 0 \\
ax^TIX - x^TAY + bx^TIX &= 0
\end{align*}
\]

Finally, taking the transpose of both sides of the second equation results in
\[
\begin{align*}
a(x^TIX)^T - (x^TAY)^T - by^TIX &= 0 \\
ax^TIX - x^TAY + bx^TIX &= 0
\end{align*}
\]

Subtracting the equations yields
\[
b(y^TIX + x^TIX) = 0
\]
or
\[
b(|Y|^2 + |X|^2) = 0
\]

Because both $|X|$ and $|Y|$ cannot be zero (for if so, $X + iY = 0$ and could not be an eigenvector), we must have $b = 0$. Therefore $\lambda$ is real.

(b) Let $X_1$ and $X_2$ be eigenvectors corresponding to $\lambda_1$ and $\lambda_2$, $\lambda_1 \neq \lambda_2$, $\lambda_1 \neq 0$. We want to show that $X_1 \cdot X_2 = 0$. Now
\[ X_1 \cdot X_2 = \frac{1}{\lambda_1} \lambda_1 X_1 \cdot X_2 = \frac{1}{\lambda_1} AX_1 \cdot X_2 = \frac{1}{\lambda_1} (AX_1)^T X_2 = \frac{1}{\lambda_1} (X_1^T A^T) X_2 \]

(By symmetry) \[= \frac{1}{\lambda_1} (X_1^T A) X_2 \]
\[= \frac{1}{\lambda_1} X_1^T (AX_2) \]
\[= \frac{1}{\lambda_1} X_1^T \lambda_2 X_2 \]
\[= \frac{\lambda_2}{\lambda_1} X_1^T X_2 \]
\[= \frac{\lambda_2}{\lambda_1} (X_1 \cdot X_2) \]

If \( \lambda_2 = 0 \), then \( X_1 \cdot X_2 = 0 \). If \( \lambda_2 \neq 0 \), then \( \frac{\lambda_2}{\lambda_1} \neq 1 \) and we must still have \( X_1 \cdot X_2 = 0 \). □

Theorem 5.4.1 tells us that we can find an orthogonal basis for each eigenspace. This may require the Gram-Schmidt process, as the next example shows.

**Example 3**  Orthogonally diagonalize

That is, diagonalize \( A \) with an orthogonal matrix \( P \).

**Solution** The characteristic polynomial is \(-\lambda^3 + 3\lambda + 2\) which has roots \(-1\) (multiplicity 2) and \(2\) (simple). To determine eigenvectors, we solve \((A - \lambda I)X = 0\):

\[
\lambda = 2: \quad \begin{align*}
-2x_1 + x_2 + x_3 &= 0 \\
x_1 - 2x_2 + x_3 &= 0 \\
x_1 + x_2 - 2x_3 &= 0
\end{align*} \Rightarrow X_1 = \begin{pmatrix} k \\ k \\ k \end{pmatrix}
\]
Since $\text{rank } (A - \lambda_2 I) = 1$, the dimension of $E(\lambda_2)$ is 2.

Looking at $X_2$ and putting $k = 1, j = 0$, we have

$$V_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

in the eigenspace. Setting $k = 0, j = 1$, we find

$$V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

in the eigenspace. Now $V_1$ and $V_2$ are not orthogonal to each other, but they are linearly independent and span the eigenspace. Using the Gram-Schmidt process on $\{V_1, V_2\}$, we find

$$\mathcal{O}_1 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \quad \text{and} \quad \mathcal{O}_2 = \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$$

as orthonormal basis vectors for the eigenspace of $\lambda_2 = -1$. Letting $\mathcal{O}_3 = X_1/|X_1|$, we obtain an orthonormal basis (for $E^3$) of eigenvectors of $A$. Choosing $P$ as

$$P = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}$$

we have

$$A = P \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} P^T$$
Now that we can orthogonally diagonalize symmetric matrices, we can consider an application to analytic geometry.

**Quadratic Forms and Conic Sections** A classical problem of analytic geometry is the following:

For the conic section centered at the origin of the $xy$ plane, described by

$$ax^2 + bxy + cy^2 = d \quad (5.4.1)$$

determine whether the conic section is an ellipse, a hyperbola, or a parabola. Graph the conic section.

A symmetric matrix can be used to describe the left-hand side of Eq. 5.4.1. In particular,

$$\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + bxy + cy^2 = d$$

Let us call

$$A = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$$

the **matrix of the conic section**.\(^{5.3}\) Making a change of basis with the orthogonal matrix $P$ which diagonalizes $A$, we write

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = P^T \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or} \quad P \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Substituting and denoting

$$\begin{pmatrix} x' \\ y' \end{pmatrix}$$

by $X'$, we have

$$X^TAX = (PX')^T A (PX') = (X'^T P^T) A (PX') = X'^T (P^T A P) X' = X'^T D X'$$

which reduces to
The last equation is easy to classify and graph in the $x'y'$ plane since it has no ``mixed'' term $x'y'$.

**Example 4** Classify $xy = 1$ and graph it.

**Solution** (We already know the graph since the equation can be rewritten as $y = 1/x$.) This will make it easy to check our answer.) The matrix of the conic section is

$$A = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

and $A$ has eigenpairs

$$\begin{pmatrix} \frac{1}{2}, \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2}, \frac{-1}{\sqrt{2}} \end{pmatrix}$$

so that

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} P^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and

$$A = P \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} P^T$$

Therefore, with the change of variables (basis)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = P^T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

which is the same as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = d$$

or

$$\lambda_1 x'^2 + \lambda_2 y'^2 = d$$
we have

\[ \frac{1}{2} x'^2 - \frac{1}{2} y'^2 = 1 \]

or

\[ \frac{x'^2}{(\sqrt{2})^2} - \frac{y'^2}{(\sqrt{2})^2} = 1 \]

Therefore, the conic section is a hyperbola. To sketch the graph, we must determine the \( x'y' \) axes. Since the point \((x, y) = (1, 1)\) gives \((x', y') = (2/\sqrt{2}, 0)\) and \((x, y) = (-1, 1)\) gives \((x', y') = (0, 2/\sqrt{2})\), the \( x' \) and \( y' \) axes are as shown in Fig. 5.4.1.

The graph of the hyperbola is also shown in Fig. 5.4.1. Note that the new axes contain the eigenvectors.

**Example 5**  Classify the conic section

\[ 2x^2 + 2xy + 2y^2 = 27 \]

and graph it.

**Solution** The matrix of the conic section is

\[ A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \]

Eigenpairs of \( A \) are

\[ \left( 3, \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right) \quad \text{and} \quad \left( 1, \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right) \]

Therefore in \( x', y' \) coordinates defined as in Example 4 (the eigenvectors are the same) we have

\[ 3x'^2 + y'^2 = 27 \]

or
which describes an ellipse. The graph of the ellipse is shown in Fig. 5.4.2. Note that the new axes contain the eigenvectors of the matrix. Also note that $x'y'$ axes are obtained by a $45^\circ$ counterclockwise rotation, which is the action of $P$. Moreover, $x'$ is defined by the first eigenvector, and $y'$ is defined by the second eigenvector.

Those who have solved these types of conic section problems in calculus realize that this linear algebra method of removing the $xy$ term is much simpler. Of course, a lot of power machinery had to be developed to get to this point.

In general, the problem of removing the $xy$ term in $ax^2 + bxy + cy^2$ is known as the problem of **diagonalizing a quadratic form**. This problem arises in many areas; statistics and physics are two. A real quadratic form in the variables $x_1, x_2, \ldots, x_n$ is a function $Q: \mathbb{R}^n \to \mathbb{R}$ given by

$$Q \left( \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right) = (x_1x_2 \cdots x_n)A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \tag{5.4.2}$$

where $A$ is in $\mathcal{M}_{nn}$. Written out, a real quadratic form in $x_1, x_2, x_3$ looks like

$$ax_1^2 + bx_2^2 + cx_3^2 + dx_1x_2 + ex_1x_3 + fx_2x_3$$

where $a$ through $f$ are real numbers. Note that each term has degree 2---hence the name **quadratic form**. Our basic theorem about diagonalization of symmetric matrices means that any real quadratic form can be diagonalized. So there are new variables $x'_1, \ldots, x'_n$ such that in the new variables $X' = P^TX$

$$Q \left( \begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix} \right) = \lambda_1 x'_1^2 + \lambda_2 x'_2^2 + \cdots + \lambda_n x'_n$$

This follows from the fact that the matrix $A$ in Eq. (5.4.2) can always be chosen as
symmetric, and symmetric matrices are orthogonally diagonalizable.

**Diagonalization in the Hermitian Case** Theorem 5.4.1 with a slight change of wording holds true for hermitian matrices.

If \( A_{n \times n} \) is hermitian, then

1. The eigenvalues are real.
2. Eigenvectors corresponding to distinct eigenvalues are orthogonal.
3. The eigenspaces of each eigenvalue have orthogonal bases. The dimension of an eigenspace corresponds to the multiplicity of the eigenvalue.
4. Matrix \( A \) is **unitarily diagonalizable**. That is, there exists a unitary matrix \( U(U^{-1} = U^*) \) such that

\[
A = UDU^* \quad \text{(thus } D = U^*AU)\]

The proofs of 1 and 2 are almost the same as in Theorem 5.4.1a and b. The difference is that \( A^* \) is used instead of \( A^T \) and in \( \mathbb{C}^n \), \( X \cdot Y = X^*Y \).

**Example 6** Can

\[
A = \begin{pmatrix}
1 & 1 - i \\
1 + i & 0
\end{pmatrix}
\]

be unitarily diagonalized? If so, perform the diagonalization.

**Solution**

\[
A^* = A^T = \begin{pmatrix}
1 & 1 - i \\
1 + i & 0
\end{pmatrix}^T = \begin{pmatrix}
1 & 1 - i \\
1 + i & 0
\end{pmatrix} = A
\]

Because \( A \) is hermitian, it can be unitarily diagonalized. Now to find the eigenpairs,

\[
\det(A - \lambda I) = (1 - \lambda)(-\lambda) - (1 - i)(1 + i) = \lambda^2 - \lambda - 2
\]

So we have \( \lambda_1 = 2 \) and \( \lambda_2 = -1 \). Eigenpairs are

\[
\left(2, \begin{pmatrix} 1 - i \\ 1 \end{pmatrix}\right) \quad \left(-1, \begin{pmatrix} 1 - i \\ -2 \end{pmatrix}\right)
\]

To find \( U \), we normalize the eigenvectors and use them for the columns of \( U \). The
normalized eigenvectors are found by calculating

\[
\left| \begin{pmatrix} 1 - i \\ 1 \end{pmatrix} \right| = \sqrt{(1 + i, 1) \left( \begin{pmatrix} 1 - i \\ 1 \end{pmatrix} \right)} = \sqrt{3}
\]

and

\[
\left| \begin{pmatrix} 1 - i \\ -2 \end{pmatrix} \right| = \sqrt{(1 + i, -2) \left( \begin{pmatrix} 1 - i \\ -2 \end{pmatrix} \right)} = \sqrt{6}
\]

(Remember that \(|X| = \sqrt{(X, X)} = \sqrt{X^*X}\).) Thus we have

\[
U = \begin{pmatrix} \frac{1 - i}{\sqrt{3}} & \frac{1 - i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \quad U^* = \begin{pmatrix} \frac{1 + i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1 + i}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{pmatrix}
\]

and

\[
\begin{pmatrix} 1 & 1 - i \\ 1 + i & 0 \end{pmatrix} = U \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} U^*
\]

When a hermitian matrix \(A\) is diagonalized, the set of orthonormal eigenvectors of \(A\) is called the set of principal axes of \(A\) and the associated matrix \(U\) is called a principal axis transformation. For a real hermitian matrix, the principal axis transformation allows us to analyze \(A\) geometrically.

**Example 7** Consider \(T_A: M_{21} \to M_{21}\) defined by

\[
T_A \left( \begin{pmatrix} a \\ b \end{pmatrix} \right) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}
\]

This can be diagonalized with

\[
U = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}
\]

so that

\[
\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}
\]
Now $U$ represents rotation of $45^\circ$ counterclockwise while $U^T$ represents rotation of $45^\circ$ clockwise. If we want to see what $T_A$ does to

\[
\begin{pmatrix}
a \\
b
\end{pmatrix}
\]

we can look at

\[
\begin{pmatrix}
1/\sqrt{2} & -1/\sqrt{2} \\
1/\sqrt{2} & 1/\sqrt{2}
\end{pmatrix}
\begin{pmatrix}
3 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
1/\sqrt{2} & 1/\sqrt{2} \\
-1/\sqrt{2} & 1/\sqrt{2}
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix}
\]

and we see that $T_A$ is a

1. Rotation of $45^\circ$ clockwise
2. Stretch of 3 in the first component and a reversal in the second component
3. Rotation of $45^\circ$ counterclockwise

We can say even more by determining what $T_A$ does to the unit circle. In the new coordinates, the unit circle is unchanged because $U$ and $U^T$ represent rotations. However, in the new coordinates we have the action of $D$ as changing the unit circle by reflecting about the line which is defined by

\[
\text{span} \left\{ \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right\}
\]

and then transforming the reflected circle to an ellipse, as shown in Fig. 5.4.3.

Finally, we note that in diagonalizing a quadratic form for a conic section, the new axes obtained from the rotation are exactly the principal axes of the matrix for the quadratic form.

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Subsections

- **PROBLEMS 5.4**