OVERALL PROPERTIES IN FIBROUS ELASTIC COMPOSITE WITH IMPERFECT CONTACT CONDITION

F.J. Sabina[1], R. Guinovart-Díaz[2], R. Rodriguez-Ramos[2],
J.C. López-Realpozo[2], J. Bravo-Castillero[2]

[1]Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México, Apartado Postal 20-726, Delegación de Álvaro Obregón, 01000 México D.F., México
[2]Facultad de Matemática y Computación, Universidad de la Habana, San Lázaro y L, Vedado, Habana 4, CP10400, Cuba

Abstract

In this contribution, the complet set of effective elastic moduli are obtained by means of the asymptotic homogenization method (AHM), for two-phase fibrous periodic composites with imperfect contact conditions of linear spring type. This work is an extension of previous reported results, where only perfect contact for elastic composite with square cells were considered. The constituents of the composites exhibit transversely isotropic properties. As validation of the present method, some numerical examples and comparisons with theoretical and experimental results verified that the present model is efficient for the analysis of composites with presence of imperfect interface. The present method can provide benchmark results for other numerical and approximate methods.

Keywords: Effective properties; imperfect contact; periodic composites; asymptotic homogenization

Introduction

Many multi-scale approaches are associated with composites where perfect interface conditions were assumed. Most of the models devoted to the micromechanical analysis

of fibre-reinforced composites consider a perfect fibre-matrix bond and suppose that stress and displacement continuity conditions are satisfied along the interface.

However, experiments show that local or partial debonding at interfaces is a rule rather than the exception in materials such as reinforced composites Hui-Zu and Tsu-Wei (1995), Rokhlin et al. (1994). The existence of a stiff region is only an idealization of the complex phenomenon that occurs at the interface where a transition zone (interphase) between the fibre and the matrix is omnipresent. This third phase results from the material fabrication process (chemical treatments of fibre surfaces, resin crystalization), Achenbach and Zhu (1989). In most fibre-reinforced composites, the fibre-matrix adhesion is imperfect; the continuity conditions for stresses and displacements are not satisfied. Thus various approaches have been used, in which the bond between the reinforcements and the matrix is modeled by an interphase with specified thickness and elastic properties different from those of the fibres and the matrix. The research works of Jasiuk et al. (1989), Jasiuk et al. (1992), Hashin (1990) are classified in this group. On the other hand, we will distinguish works supposing the interphase as a boundary layer (no thickness specified) through which it is assumed that the stresses are continuous but not the displacements. This approach has been employed by several authors, like, Hashin (1991a,b), Hashin (2002).

According to preceding observations the selection of an appropriate model to represent the interphase is important. The micromechanical models are able to answer these requirements. Their purpose is to determine the response of fibre-reinforced composites by the knowledge of the fibre and the matrix behaviors.

Some more recently researches are studied the influence of the interphase/interface in the composites. Caporale et al. (2006) investigated the behavior of unidirectional fiber reinforced composites with imperfect interfacial bonding by using the finite element method. In this work, an interfacial failure model is implemented by connecting the fibers and the matrix at the finite element nodes by normal and tangential brittle-elastic springs. Furthermore, Bisegna and Caselli (2008), Artioli et al. (2010) obtained closed-form expression for the homogenized longitudinal shear moduli of a linear elastic composite material reinforced by long, parallel, circular fibres with a periodic arrangement. An imperfect linear elastic fibre-matrix interface is allowed.

The aim of this paper is to point out the effects of the fibre-matrix interface on the mechanical properties of elastic two phase composite materials based on the analytical expressions obtained by micromechanical model of cell assembly, in particular, the two

scale asymptotic homogenization method, proposed by Bakhalov and Panasenko (1989), Sanchez-Palencia (1980), Pobedria (1984). This work is an extension of previous reported results, where only perfect contact for elastic composite with square cells were considered. In this contribution, the complet set of effective elastic moduli by means of the asymptotic homogenization method (AHM), for two-phase fibrous periodic composites with imperfect contact conditions of linear spring type are obtained. As validation of the present method, the Hills's universal relations are satisfied and some numerical examples, comparisons with theoretical and experimental results verified that the present model is efficient for the analysis of composites with presence of imperfect interface.

Statement of the problem. Local problems based on asymptotic homogenization method

The mechanical behavior of imperfect interface is modeled via a layer of mechanical springs of zero thickness. The spring constants are the measures for the magnitude of the associated continuities on Γ, where K_n, K_s and K_t are spring constant type material parameters which have dimension of stress divided by length. We shall call these constants the *interface parameters*. It is seen that infinite values of the parameters imply vanishing of displacement jumps and therefore perfect interface conditions. At the other extremity zero values of the parameters implies vanishing of interface tractions and therefore disband. Any finite position values of the interface parameters define an imperfect interface. This may be due to the presence of an interphase but also due to interface bond deterioration caused for reasons such as fatigue damage or environmental and chemical effects. Within this approach the composite is modeled as a two phase material with imperfect interface conditions.

The vanishing value of K_n, K_s and K_t corresponds to pure debonding (normal perfect debonding), in-plane pure sliding, and out-of-plane pure sliding, respectively. The status of the mechanical bonding is completely determined by appropriate values of these constants. For large enough values of the constants, the perfect bonding interface is achieved.

Using the vector notation and defining the spring stiffness matrix, the mechanic displacement and the traction vectors in the following manner

$$\mathbf{u} = \begin{pmatrix} u_t \\ u_s \\ u_n \end{pmatrix}, \ \mathbf{T} = \begin{pmatrix} T_t \\ T_s \\ T_n \end{pmatrix}, \ \mathbf{K} = \begin{pmatrix} \widetilde{K}_t & 0 & 0 \\ 0 & \widetilde{K}_s & 0 \\ 0 & 0 & \widetilde{K}_n \end{pmatrix}, \tag{1}$$

the mechanical imperfect condition (Hashin 1990, Shodja et al. 2006) in general may be expressed as

$$\mathbf{T}^{(1)} + \mathbf{T}^{(2)} = 0, \quad \mathbf{T}^{(\gamma)} = (-1)^{\gamma+1}\,\mathbf{K}\,\|\mathbf{u}\|, \ \text{on } \Gamma. \tag{2}$$

In these relations $\|\bullet\|$ indicates the jump in the quantity at the common interface between the fiber and the matrix denoted by Γ; $\mathbf{n}$ is the outward unit normal on Γ; u_t, u_s, u_n are the tangential and normal components of the mechanic displacement vector; T_t, T_s, T_n are the tangential and normal components of the traction vector $\mathbf{T}$ ($T_i = \sigma_{ij} n_j$). The superscripts (γ), $\gamma = 1, 2$ denote the matrix and fiber respectively.

We can consider this representation to be preferred over the three phase description for some reasons, for instance: a) It is more general in the sense that the three phase description is a special case of eqn. (2) since, the interface parameters can be expressed in terms of interphase properties. Furthermore, it is applicable also when an interphase cannot be defined or identified; b) it characterizes imperfect interface in terms of a small number of parameters which may hopefully be determined experimentally; c) it is mathematically simpler and more transparent.

A two-phase uniaxial reinforced material is considered here in which fibers and matrix have transversely isotropic elastic properties; the axis of transverse symmetry coincides with the fiber direction, which is taken as the OX_3 axis. The fibers cross-section is circular. Moreover, the fibers are periodically distributed without overlapping in directions parallel to the Ow_1- and Ow_2-axis, where $w_1 \neq 0$ and $w_2 \neq 0$ $(w_2 \neq \lambda w_1,\ \lambda \in \mathbb{R})$ are two complex numbers which define the square periodic cell of the two-phase composite. (see, Fig. 1). Therefore the composite Ω consists of a square array of identical circular cylinders embedded in a homogeneous medium (Fig.1)

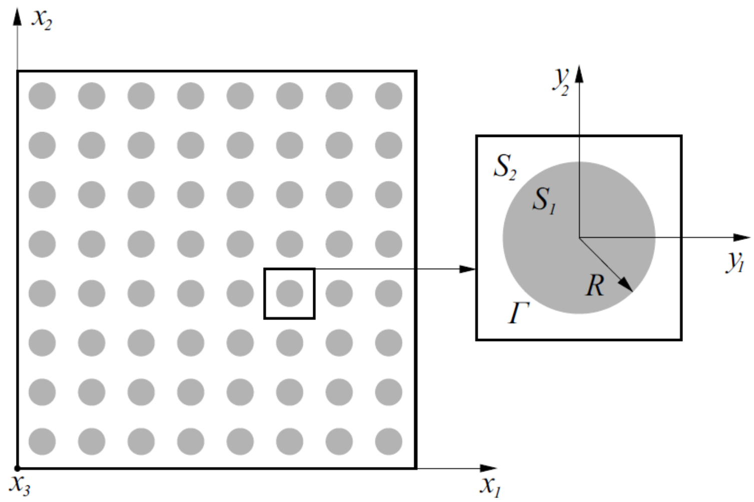

Fig.1 The cross section of a quadratic and periodic array of circular fibers.

The overall properties of the above periodic medium are sought using the well-known asymptotic homogenization method. Then, it follows that in terms of the fast variable $\mathbf{y}$, the appropriate periodic unit cell S is taken as a regular square in the $y_1 y_2$-plane so that $S = S_1 \cup S_2$ with $S = S_1 \cap S_2 = \emptyset$, where the domain S_2 is occupied by the matrix and its complement S_1 (fiber) is considered by a circle of radius R and center at the origin O (Fig. 1). The common interface between the fiber and the matrix is denoted by Γ. The fiber and matrix associated quantities are also referred below by means of super-indices in brackets (1) and (2), respectively.

Using the conventional indicial notation in which repeated subscripts are summed over the range of $i, j, k, l = 1, 2, 3$, the constitutive equation is

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}, \tag{3}$$

where $\sigma_{ij}, \varepsilon_{ij}$ are the stress and strain tensors, respectively. The quantities C_{ijkl} are components of the elastic stiffness tensor. The elastic equilibrium equation as the body forces are absent is,

$$\sigma_{ij,j} = 0, \text{ in } \Omega, \tag{4}$$

where the subscript comma denotes partial differentiation. The gradient equations, which are the strain-displacement equations are

$$\varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right), \tag{5}$$

where u_i are the components of the mechanic displacement.

Substituting (3) and (5) into (4) we obtain a system of partial differential equations with coefficients rapidly oscillating

$$\left(C_{ijkl}(\mathbf{y})u^{\varepsilon}_{k,l}(\mathbf{x})\right)_{,j} = 0, \text{ in } \Omega. \tag{6}$$

Equation (6) represents a system of equations for finding u_i. For a complete solution, it is necessary to assign suitable boundary and interface conditions, for instance

$$u^{\varepsilon}_i = 0; \quad \sigma^{\varepsilon}_{ij}n_j = S^0_i, \quad \text{on } \partial\Omega, \tag{7}$$

where u^0_i, S^0_i are the prescribed displacement, force on the boundary of the composite. In order to study the imperfect contact conditions (2) we consider the relations between the mechanic displacement and the traction vectors (1) with Cartesian mechanic displacement (u_i) and the traction (T_i) vectors respectively by the following expressions

$$\begin{pmatrix} u_n \\ u_t \\ u_s \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \begin{pmatrix} T_n \\ T_t \\ T_s \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}. \tag{8}$$

On the interface Γ, the functions $\mathbf{u}(\mathbf{x})$ satisfies a particular case of non-ideal contact conditions (2), taking $\widetilde{K}_n \to \infty$; it is equivalent to the continuity of the displacement in the normal direction, $\|u_n\| = 0$ and $0 < \widetilde{K}_t < \infty$, $0 < \widetilde{K}_s < \infty$. Thus the expression (2) on Γ, can be rewritten in the following indicial form,

$$\mathbf{T}^{(1)} + \mathbf{T}^{(2)} = 0, \tag{9}$$

$$\|u_n\| = 0, \tag{10}$$

$$T^{(\gamma)}_t = (-1)^{\gamma+1}\widetilde{K}_t \|u_t\|, \tag{11}$$

$$T^{(\gamma)}_s = (-1)^{\gamma+1}\widetilde{K}_s \|u_s\|. \tag{12}$$

By means of the AHM it is possible to obtain an asymptotic solution of the boundary-value problem (6)-(12) as $\varepsilon \to 0$. The solution is sought in the form of a series in powers of ε with coefficients depending on both the variables $\mathbf{x}$ and $\mathbf{y} = \mathbf{x}/\varepsilon$ treated as independent; they are referred as the slow or macroscopic and fast or microscopic variables, respectively, where ε = l/L is a small dimensionless parameter and L is a linear dimension of the body. Here, the solution is explicitly posed as

$$\mathbf{u}^{\varepsilon}(\mathbf{x}) = \mathbf{u}^{(0)}(\mathbf{x}) + \varepsilon \mathbf{u}^{(1)}(\mathbf{x},\mathbf{y}) + O(\varepsilon^2)\,, \tag{13}$$

where $\mathbf{u}^{(0)}$ satisfy the homogenized system of differential equations

$$C^{*}_{ijlk} u^{(0)}_{l,kj} = 0. \quad \text{on} \quad \Omega\,, \tag{14}$$

and the asterisk denotes the overall elastic properties. The term $\mathbf{u}^{(1)}$ represent a correction of the $\mathbf{u}^{(0)}$. The function $\mathbf{u}^{(1)}$ is found in combination of the function $\mathbf{M}(\mathbf{y})$ (solution of the local problems) and the partial derivatives of the function $\mathbf{u}^{(0)}$.

Then, the original constitutive relations with rapidly oscillating material coefficients (6) are transformed into equivalent system (14) with constant coefficients $\mathbf{C}^{*}$ which represent the elastic properties of an equivalent homogeneous medium. They are called effective coefficients of the composite Ω.

The main problem to obtain average coefficients is to find the periodic solutions of six ${}_{pq}L$ local problems on S in terms of the fast variable $\mathbf{y}$, where $p,q = 1,2,3$. Each local problem uncouples into sets of equations, i.e. plane-strain and antiplane-strain systems. In the following Table, the correspondence between the effective properties and the local problems is shown.

${}_{11}L$	${}_{22}L$	${}_{33}L$	${}_{23}L$	${}_{13}L$	${}_{12}L$
C^{*}_{1111}	C^{*}_{1122}	C^{*}_{1133}	0	0	0
C^{*}_{2211}	C^{*}_{2222}	C^{*}_{2233}	0	0	0
C^{*}_{3311}	C^{*}_{3322}	C^{*}_{3333}	0	0	0

0	0	0	C^*_{1313}	0	0
0	0	0	0	C^*_{2323}	0
0	0	0	0	0	C^*_{1212}

Table: Effective properties related to the local problems

In the present work, the ${}_{pq}L$ problem consist to find the displacements ${}_{pq}M^{(\gamma)}(\mathbf{y})$ in S_γ, $\gamma = 1, 2$ (double periodic functions with periods $w_1 = 1, w_2 = b\,e^{i\Theta}$; $b > 0$) as solution of the following system of partial differential equations

$$ {}_{pq}\sigma^{(\gamma)}_{i\delta,\delta} = 0 \;\text{ in } S_\gamma, \tag{15}$$

where

$$ {}_{pq}\sigma^{(\gamma)}_{i\delta} = C^{(\gamma)}_{i\delta k\lambda}\; {}_{pq}M^{(\gamma)}_{k,\lambda}, \tag{16}$$

the comma notation denotes a partial derivative relative to the y_δ component, i.e., $U_{,\delta} \equiv \partial U/\partial y_\delta$; the summation convention is also understood for Greek indices, which run from 1 to 2; no summation is carried out over upper case indices, whether Latin on Greek.

Thus the expression (9)-(12) on Γ for the ${}_{pq}L$ problem can be expressed in the following indicial form,

$$ {}_{pq}\mathbf{T}^{(1)} + {}_{pq}\mathbf{T}^{(2)} = 0, \tag{17}$$

$$ \left\| {}_{pq}M_n \right\| = 0, \tag{18}$$

$$ {}_{pq}T^{(\gamma)}_t = (-1)^{\gamma+1}\, \widetilde{K}_t \left\| {}_{pq}M_t \right\|, \tag{19}$$

$$ {}_{pq}T^{(\gamma)}_s = (-1)^{\gamma+1}\, \widetilde{K}_s \left\| {}_{pq}M_s \right\|, \tag{20}$$

where M_t, M_s, M_n and T_t, T_s, T_n have the same meaning as (1) but adequate to ${}_{pq}L$ problems. To assure the only one solution of the ${}_{pq}L$ problems, the functions also satisfy

that $\left\langle {}_{pq}M_k \right\rangle = 0,$ where the angular brackets define the volume average per unit length over the unit periodic cell, that is $\langle F \rangle = \int_{|S|} F(\mathbf{y})\, d\mathbf{y}.$ The symmetry between the indices p and q shows right away that at most six problems need to be considered. Once the local problems are solved, the homogenized moduli C^*_{ijpq} may be determined by using the following formulae:

$$C^*_{ijpq} = \left\langle C_{ijpq} + C_{ijkl}\;{}_{pq}M_{k,l} \right\rangle. \qquad (21)$$

The potential method of complex variables $z = y_1 + iy_2,\ (y_1, y_2) \in S$ and the properties of doubly periodic Weierstrass $\wp(z) = \frac{1}{z^2} + \sum_{s,t}{}' \left\{ \frac{1}{(z - P_{st})^2} - \frac{1}{P_{st}^2} \right\}$ and related functions (Z- function $\zeta(z) = -\wp'(z)$ and Natanzon´s function $Q(z)$) are used for the solution of the local problems (15)-(20). Hence, the non-zero solution ${}_{pq}M_k^{(\alpha)}$ in S_α of the problem (15)-(20) must be found among doubly periodic functions of periods $w_1 = 1,\ w_2 = i$ (see Fig. 1). Each local problem (15)-(20) uncouples into sets of equations. An in-plane strain system for ${}_{pq}M_\lambda^{(\alpha)},\ \lambda = 1, 2$ and an out-of-plane strain Laplace’s equation for ${}_{pq}M_3^{(\alpha)}$ has to be solved. Then the solution of the in-plane (out-of-plane) strain problems involves the determination of the in-plane (out-of-plane) displacements, strains and stresses over each phase S_α of the composite. Due to the non-vanishing components of the elastic tensors $C^{(\alpha)}_{ijpq},$ the only non-homogeneous problems, that have a non-zero solution, correspond to the four in-plane strain problems ${}_{jj}L$ and ${}_{12}L$, and the two out-of-plane strain ones ${}_{23}L$ and ${}_{13}L.$ In that way the solutions of both (in-plane and out-of-plane) local problems (Guinovart-Díaz et al., 2001, Rodriguez-Ramos et al., 2001) (six ${}_{pq}L$ problems need to be considered due to the symmetry between p and q) lead to obtain the average coefficients of the composite given in Fig. 1.

Solution of plane problem $_{jj}L$ and the effective coefficients $C^*_{1111}, C^*_{1122}, C^*_{1133}$ and C^*_{3333}

For the sake of clarity, in this section let $U^{(\gamma)} = {}_{jj}M^{(\gamma)}$ and the jj presubindices are dropped from all relevant quantities. From the equations (15)-(20) using the relation (8) we have

$$\sigma^{(\gamma)}_{\alpha\delta,\delta} = 0 \quad \text{in} \quad S_\gamma; \text{ where } \sigma^{(\gamma)}_{\alpha\delta} = C^{(\gamma)}_{\alpha\delta\beta\lambda}\, U^{(\gamma)}_{\beta,\lambda}, \tag{22}$$

$$\left\| U^{(\gamma)}_\alpha \right\| n_\alpha = 0 \quad \text{on} \quad \Gamma, \tag{23}$$

$$\left\| \sigma^{(\gamma)}_{\alpha\delta} n_\delta \right\|_1 = -\left\| C_{\alpha\delta jj} \right\|_1 n_\delta \quad \text{on} \quad \Gamma, \tag{24}$$

$$\left[\sigma^{(\gamma)}_{11} - \sigma^{(\gamma)}_{22} + \left(C^{(\gamma)}_{11jj} - C^{(\gamma)}_{22jj}\right)\right] n_1 n_2 + \sigma^{(\gamma)}_{12}\left(n_2^2 - n_1^2\right) = (-1)^{1+\gamma}\,\widetilde{K}_t\left[\left\| U^{(\gamma)}_2 \right\| n_1 - \left\| U^{(\gamma)}_1 \right\| n_2\right] \text{ on}\,\Gamma, \tag{25}$$

where

$$\begin{aligned} \sigma^{(\gamma)}_{11} &= (k_\gamma + m_\gamma)U^{(\gamma)}_{1,1} + (k_\gamma - m_\gamma)U^{(\gamma)}_{2,2}, \\ \sigma^{(\gamma)}_{22} &= (k_\gamma - m_\gamma)U^{(\gamma)}_{1,1} + (k_\gamma + m_\gamma)U^{(\gamma)}_{2,2}, \\ \sigma^{(\gamma)}_{12} &= m_\gamma (U^{(\gamma)}_{1,2} + U^{(\gamma)}_{2,1}), \end{aligned} \tag{26}$$

where $k = \left(C_{1111} + C_{1122}\right)/2$ is the plane-strain bulk modulus for lateral dilatation without longitudinal extension; $m = C_{1212} = \left(C_{1111} - C_{1122}\right)/2$ is the rigid modulus for longitudinal uniaxial straining.

It can be recognized that the structure of Eqs. (22) and (26) is one of plane-strain isotropic elasticity except that the usual Lame constants λ and μ are here identified with $k_\gamma - m_\gamma$ and m_γ, respectively. The method of complex variables in terms of two harmonic functions $\varphi_\gamma(z)$ and $\psi_\gamma(z)$ and the Kolosov-Muskhelishvili complex potentials are applicable. The potentials are related to the displacement and stress components by means of the classical formulae

$$2m_\gamma (U^{(\gamma)}_1 + iU^{(\gamma)}_2) = \kappa_\gamma \varphi_\gamma(z) - z\overline{\varphi}_\gamma(z) - \overline{\psi}_\gamma(z), \tag{27}$$

$$\sigma^{(\gamma)}_{11} + \sigma^{(\gamma)}_{22} = 2(\varphi'_\gamma(z) + \overline{\varphi}'_\gamma(z)), \tag{28}$$

$$\sigma^{(\gamma)}_{22} - \sigma^{(\gamma)}_{11} + 2i\sigma^{(\gamma)}_{12} = 2(\overline{z}\varphi'_\gamma(z) + \psi'_\gamma(z)), \tag{29}$$

where the prime denotes a derivative with respect to z, the over bar a complex conjugate and $\kappa_\gamma = 3 - 4\nu_\gamma^T$, here ν_γ^T is the transverse Poisson's ratio,

The non-ideal contact conditions (23)-(25) are transformed according to the Kolosov-Muskhelishvili complex potentials by the formulae (28)-(29) in the following form

$$\begin{aligned}&\chi_m\left[\chi^{(1)}\varphi_1(z) - z\overline{\varphi_1'}(z) - \overline{\psi_1}(z)\right] - \left[\kappa_2\varphi_2(z) - z\overline{\varphi_2'}(z) - \overline{\psi_2}(z)\right] + \\ &\chi_m\left[\chi^{(1)}\overline{\varphi_1}(z) - \bar{z}\varphi_1'(z) - \psi_1(z)\right]e^{2i\theta} - \left[\kappa_2\overline{\varphi_2}(z) - \bar{z}\varphi_2'(z) - \psi_2(z)\right]e^{2i\theta} = 0,\end{aligned} \tag{30}$$

$$z\overline{\varphi_1'}(z) + \overline{\psi_1}(z) + \varphi_1(z) + \bar{z}\gamma_1(\beta) + z\gamma_2(\beta) = z\overline{\varphi_2'}(z) + \overline{\psi_2}(z) + \varphi_2(z), \tag{31}$$

$$\begin{aligned}&2R\chi_m\left[z\overline{\varphi_2''}(z) + \overline{\psi_2'}(z)\right]e^{-i\theta} - 2R\chi_m\left[\bar{z}\varphi_2''(z) + \psi_2'(z)\right]e^{3i\theta} - 2\gamma_3(\beta)R\chi_m\left(e^{3i\theta} - e^{-i\theta}\right) = \\ &= -K_t\left\{\begin{aligned}&\chi_m\left[\chi^{(1)}\varphi_1(z) - z\overline{\varphi_1'}(z) - \overline{\psi_1}(z)\right] - \left[\chi^{(2)}\varphi_2(z) - z\overline{\varphi_2'}(z) - \overline{\psi_2}(z)\right] - \\ &\chi_m\left[\chi^{(1)}\overline{\varphi_1}(z) - \bar{z}\varphi_1'(z) - \psi_1(z)\right]e^{2i\theta} - \left[\chi^{(2)}\overline{\varphi_2}(z) - \bar{z}\varphi_2'(z) - \psi_2(z)\right]e^{2i\theta}\end{aligned}\right\},\end{aligned} \tag{32}$$

where

$$\chi_m = m_2/m_1,\ \ 2\gamma_{1j} = C_{22jj}^{(1)} - C_{22jj}^{(2)} + C_{11jj}^{(2)} - C_{11jj}^{(1)},\ \ 2\gamma_{2j} = C_{11jj}^{(1)} - C_{11jj}^{(2)} + C_{22jj}^{(1)} - C_{22jj}^{(2)}, \tag{33}$$

and $K_t = \tilde{K}_t R / m_1$ is a dimensionless parameter.

The complex potentials $\varphi_\gamma(z)$ and $\psi_\gamma(z)$ are looked for the periodic cell that contains the origin of coordinates in the following form

$$\begin{aligned}\varphi_1(z) &= \frac{a_0}{R}z + \varsigma(z)Ra_1 + \sum_{k=3}^{\infty}{}^{*}\sum_{m,n}^{\infty}\left(\frac{R}{z-\beta_{mn}}\right)^k a_k, \\ \psi_1(z) &= \frac{z}{R}b_0 + \varsigma(z)Rb_1 + Q(z)Ra_1 + \sum_{k=3}^{\infty}{}^{*}\sum_{m,n}^{\infty}\left(\frac{R}{z-\beta_{mn}}\right)^k b_k + \sum_{k=3}^{\infty}{}^{*}\sum_{m,n}^{\infty}\frac{\overline{\beta_{mn}}kR^k}{(z-\beta_{mn})^{k+1}}a_k, \\ \varphi_2(z) &= \sum_{k=1}^{\infty}{}^{*}\left(\frac{z}{R}\right)^k c_k, \\ \psi_2(z) &= \sum_{k=1}^{\infty}{}^{*}\left(\frac{z}{R}\right)^k d_k.\end{aligned} \tag{34}$$

For these local problems, we propose the doubly periodic potentials

$$\varphi_1(z) = a_0 z + \sum_{k=1}^{\infty}{}^{0} \frac{a_k \zeta^{(k-1)}(z)}{(k-1)!},$$
$$\psi_1(z) = b_0 z + \sum_{k=1}^{\infty}{}^{0} \frac{b_k \zeta^{(k-1)}(z)}{(k-1)!} + \sum_{k=1}^{\infty}{}^{0} \frac{a_k Q^{(k-1)}(z)}{(k-1)!}, \tag{35}$$

for the matrix phase and

$$\varphi_2(z) = \sum_{k=1}^{\infty}{}^{0}\ c_k z^k,$$
$$\psi_2(z) = \sum_{k=1}^{\infty}{}^{0}\ d_k z^k, \tag{36}$$

for the fiber phase.

In the foregoing definitions $\zeta^{(k-1)}(z)$ denotes the doubly–periodic $(k-1)$-th derivative of the Weierstrass Zeta quasi–periodic function $\zeta(z)$, $Q^{(k-1)}(z)$ is the doubly–periodic $(k-1)$-th derivative of Natanzon's quasi–periodic function, and the symbol $\sum^{0}$ indicates a sum over the indices k = 1, 3, 5, . . ., meanwhile coefficients a_0, b_0, a_k, b_k, c_k and d_k (k = 1, 3, 5, . . .) are all real and undefined.

Next, by taking into consideration the double periodicity of matrix displacements $U^{(1)}$, by means of Legendre's relation and the properties of the doubly periodic functions, it is found that coefficients a_0 and b_0 are

$$a_0 = \frac{V_2}{\kappa_1 - 1} b_1, \quad b_0 = \left(\kappa_1 + \frac{5S_4}{\pi^2}\right) V_2 a_1. \tag{37}$$

Replacing (35)-(36) into (30)-(32) the following relations between the unknown constants of the above expansions are obtained

$$b_{p+2} = p a_p - \sum_{k=1}^{\infty}{}^{0} \eta_{k(p+2)} a_k + c_{p+2}, \tag{38}$$

$$(p+2) c_{p+2} + d_p = a_p + b_0 \delta_{1p} + R\gamma_{1j}\delta_{1p} + \eta_{1p} b_1 + \sum_{k=1}^{\infty}{}^{0} (p+2)\eta_{k(p+2)} a_k + \eta_{k+2p} b_{k+2} + kR^{p+k} C_{p+k}^{p} T_{p+k} a_k, \tag{39}$$

$$c_{p+2} = -\frac{D_p}{E_p} K_t \chi_m (\kappa_1 + 1) a_p - \frac{B_p}{E_p} K_t \chi_m (\kappa_1 + 1) \sum_{k=1}^{\infty}{}^{0} \eta_{k(p+2)} a_k, \tag{40}$$

$$b_1 = F\sum_{k=1}^{\infty}{}^{0}\eta_{k1}a_k - PR\gamma_2, \tag{41}$$

$$c_1 = \frac{\chi_m}{2\alpha_0}\left[(\kappa_1+1)\sum_{k=1}^{\infty}{}^{0}\eta_{k1}a_k + (1-V_2)R\gamma_{2j}\right], \tag{42}$$

$$F = \frac{1+\chi_m\kappa_1-\chi_m-\kappa_2}{\alpha_0}, \quad P = \frac{\kappa_2-1}{2\alpha_0}, \alpha_0 = \chi_m(1-V_2)+(\kappa_2-1)\left(\frac{V_2}{\kappa_1-1}+\frac{1}{2}\right),$$

where δ_{ik} is the Kronecker's delta function, C_k^n is the binomial coefficient, V_1, V_2 are the area of the matrix and fiber respectively, $V_1+V_2=1$, $V_2=\pi R^2$. The parameters involved in this expression are listed as follows

$$\eta_{kp} = -k\frac{(k+p-1)!}{k!\ p!}S_{k+p}R^{k+p}, \quad \alpha_0 = \chi_m[1-V_2]+(\kappa_2-1)\left(\frac{V_2}{\kappa_1-1}+\frac{1}{2}\right),$$

$$S_{n+k} = \sum_{p,q}\frac{1}{(p+iq)^{n+k}},\ T_{n+k} = \sum_{p,q}\frac{p-iq}{(p+iq)^{n+k+1}}, \quad p^2+q^2\neq 0, p,q - \text{integer numbers},$$

$$A_p = \left[\chi_m(p+1)-K_t(\chi_m-1)\right](p+2), \quad B_p = K_t(1-\chi_m)+\chi_m p,$$

$$C_p = K_t(\chi_m+\kappa_2)+\chi^*(p+2)(p+1), \quad D_p = \chi_m p, \quad E_p = A_pD_p - C_pB_p.$$

The above expression (38)-(42) depends on the unknown parameter a_k which can be calculated from the following system of algebraic equations

$$\begin{aligned} a_p &+ B_1^1V_2\left(\kappa_1+\frac{5S_4}{\pi^2}\right)a_1\delta_{1p} + \\ &+\sum_{k=1}^{\infty}{}^{o}\left[\begin{aligned} B_p^1G_{kp} - B_p^1C_k^0r_{kp} + B_p^1F\eta_{1p}\eta_{k1} + \\ + B_p^0\left(A_p^0\right)^{-1}\eta_{k\ p+2} - B_p^1D_p^0\eta_{k+2\,p}\end{aligned}\right]a_k = \eta_{1p}B_p^1PR\gamma_{2j} + E\gamma_{1j}R\delta_{1p}, \end{aligned} \tag{43}$$

where $p = 1,3,5,\ldots$,

$$A_p^0 = 1 + \chi_m \kappa_1 + K_t \chi_m (1+\kappa_1)(\chi_m + \kappa_2)\frac{D_p}{E_p},\ B_p^0 = \chi_m (1+\kappa_1)\left[1 + (\chi_m + \kappa_2)\frac{B_p}{E_p} K_t\right],$$

$$C_p^0 = 1 + K_t \chi_m (1+\kappa_1)\frac{B_p}{E_p},\quad D_p^0 = \frac{D_p}{E_p} K_t \chi_m (1+\kappa_1),\ B_p^1 = (1-\chi_m)\left(A_p^0\right)^{-1}, E = -\left(A_1^0\right)^{-1},$$

$$r_{kp} = \sum_{i=3}^{\infty}{}^{0} \eta_{ki}\eta_{ip},\ G_{kp} = (p+2)\eta_{k(p+2)} + k\eta_{k+2\,p} + kR^{p+k} C_{p+k}^{p} T_{p+k}.$$

It should be noticed that equation (43) represents an infinite linear system from which we can calculate the sought coefficients a_p. The unknown coefficients a_p can be obtained as in Rodriguez-Ramos et al. (2001). We present here only the most important results in order to obtain the effective properties. The residue a_1 of the function $\varphi_1(z)$ is

$$a_1 = \frac{E\gamma_{1j}R}{1 + H^+ - \mathfrak{U}_1 \mathfrak{U}^{-1} \mathfrak{U}_2}, \tag{44}$$

where $H^+ = B_1^1 V_2 \left(\chi^{(1)} + 5S_4/\pi^2\right) + B_1^1 G_{11} - B_1^1 C_1^0 r_{11} + B_1^0 \eta_{13}/A_1^0 - B_1^1 D_1^0 \eta_{31}$ and $\mathfrak{U}$, $\mathfrak{U}_1$ and $\mathfrak{U}_2$ denotes a matrix and two vectors of infinite order respectively and $\mathfrak{U}^{-1}$ is the inverse of the matrix $\mathcal{U}$, given by the following expressions

$$\mathcal{U}[u_{ts}] = \left[\delta_{ts} + B_{4s+1}^1 G_{4t+1\,4s+1} - B_{4s+1}^1 C_{4t+1}^0 r_{4t+1\,4s+1} + B_{4s+1}^0 \eta_{4t+1\,4s+3}/A_{4s+1}^0 - B_{4s+1}^1 D_{4s+1}^0 \eta_{4t+3\,4s+1}\right],$$
$$\mathfrak{U}_1[u_s] = \left[B_{4s+1}^1 G_{1\,4s+1} - B_{4s+1}^1 C_1^0 r_{1\,4s+1} + B_{4s+1}^0 \eta_{1\,4s+3}/A_{4s+1}^0 - B_{4s+1}^1 D_{4s+1}^0 \eta_{3\,4s+1}\right],$$
$$\mathfrak{U}_2[u_t] = \left[B_1^1 G_{4t+1\,1} - B_1^1 C_{4t+1}^0 r_{4t+1\,1} + B_1^0 \eta_{4t+1\,3}/A_{4s+1}^0 - B_1^1 D_1^0 \eta_{4t+3\,1}\right].$$

Once the solution of the previous system (43) is obtained, the coefficient c_1 in (42) can be written in the following form

$$c_1 = \frac{\chi_m \gamma_{2j} R}{2\alpha_0}\left[P(\kappa_1 + 1)\aleph_1 \mathcal{Z}^{-1} \aleph_2 + (1 - V_2)\right], \tag{45}$$

where again the capital letters $\mathcal{Z}$, $\aleph_1$ and $\aleph_2$ denotes a matrix and two vectors of infinite order respectively, with the following components

$$\mathcal{Z}(z_{ts}) = \begin{pmatrix} \delta_{ts} + B_{4s-1}^1 G_{4t-1\,4s-1} - B_{4s-1}^1 C_{4t-1}^0 r_{4t-1\,4s-1} + B_{4s-1}^1 F \eta_{4t-1\,1}\eta_{1\,4s-1} + \\ + B_{4s-1}^0 \eta_{4t-1\ 4s+1} / A_{4s-1}^0 - B_{4s-1}^1 D_{4s-1}^0 \eta_{4t+1\,4s-1} \end{pmatrix},$$

$$\aleph_1(n_s) = \left(B_{4s-1}^1 \eta_{1\,4s-1}\right),$$

$\aleph_2(n_t) = (\eta_{4t-11})$, $t,s = 1,2,3,\ldots.$

In order to obtain analytic expressions of the effective properties $C^*_{1111}, C^*_{1122}, C^*_{1133}$ and C^*_{3333}, is necessary to apply the Green's theorem to (21) on the interface Γ and using the condition (27) and the orthogonality of the trigonometric functions we get

$$C^*_{11\,jj} + C^*_{22\,jj} = \left\langle C_{11\,jj} + C_{22\,jj} \right\rangle - 2\pi R \frac{\| k \| (\kappa_2 - 1) c_1}{\chi_m m_1}, \tag{46}$$

$$C^*_{22\,jj} - C^*_{11\,jj} = \left\langle C_{22\,jj} - C_{11\,jj} \right\rangle + 2\pi R \left[\left(\chi^{(1)} + 1 \right) a_1 + \gamma_{1\,j} \right], \tag{47}$$

$$C^*_{33\,jj} = \left\langle C_{33\,jj} \right\rangle - \pi R \frac{\| C_{13} \| (\kappa_2 - 1) c_1}{\chi_m m_1}. \tag{48}$$

The following analytical expressions of the effective properties are obtained replacing the previous equalities (33), (44) and (45) into (46)-(48)

$$C^*_{1111} = \left\langle C_{1111} \right\rangle + V_2 \left(\| k \|^2 \Lambda_1 + \| m \| \Lambda_2 \right), \tag{49}$$

$$C^*_{1122} = \left\langle C_{1122} \right\rangle + V_2 \left(\| k \|^2 \Lambda_1 - \| m \| \Lambda_2 \right), \tag{50}$$

$$C^*_{1133} = C^*_{2233} = \left\langle C_{1133} \right\rangle + V_2 \| k \| \| C_{1133} \| \Lambda_1, \tag{51}$$

$$C^*_{3333} = \left\langle C_{3333} \right\rangle + V_2 \| C_{1133} \|^2 \Lambda_1, \tag{52}$$

where

$$\Lambda_1 = \frac{(1-\kappa_2)}{2m_1\alpha_0} \left[V_1 + \frac{\kappa_2 - 1}{2\alpha_0} (\kappa_1 + 1) \aleph_1 \mathcal{Z}^{-1} \aleph_2 \right], \Lambda_2 = \left[\frac{E(\kappa_1 + 1)}{1 + H^+ - \mho_1 \mho^{-1} \mho_2} + 1 \right].$$

The plane problem $_{12}\mathrm{L}$

Let $\mathrm{U}^{(\gamma)} = {}_{12}\mathrm{M}^{(\gamma)}$, from the equations (15)-(20) and using the relation (8), the local problem is stated as follows

$$\sigma^{(\gamma)}_{\alpha\delta,\delta} = 0 \quad \text{in} \quad S_\gamma; \text{ where } \sigma^{(\gamma)}_{\alpha\delta} = \mathrm{C}^{(\gamma)}_{\alpha\delta\beta\lambda}\, \mathrm{U}^{(\gamma)}_{\beta,\lambda}, \tag{53}$$

$$\left\| U^{(\gamma)}_\alpha \right\| n_\alpha = 0 \quad \text{on} \quad \Gamma, \tag{54}$$

$$\left\| \sigma_{\alpha\delta}^{(\gamma)} n_\delta \right\| = -\left\| C_{\alpha\delta 12} \right\| n_\delta \quad \text{on} \quad \Gamma\,, \tag{55}$$

$$\left[\sigma_{11}^{(\gamma)} - \sigma_{22}^{(\gamma)} \right] n_1 n_2 + \left(\sigma_{12}^{(\gamma)} + C_{1212}^{(\gamma)} \right) \left(n_2^2 - n_1^2 \right) = \left(-1 \right)^{1+\gamma} \widetilde{K}_t \left[\left\| U_2^{(\gamma)} \right\| n_1 - \left\| U_1^{(\gamma)} \right\| n_2 \right] \quad \text{on}\,\Gamma. \tag{56}$$

Once the plane problems ${}_{jj}\mathrm{L}$ have been solved, it is easier to solve the plane problem ${}_{12}\mathrm{L}$. This can be realized by means of the next relationship between the Kolosov-Muskhelishvili potentials for plane local problems,

$${}_{12}\varphi_\gamma(\mathrm{z}) = \mathrm{i}\left({}_{jj}\varphi_\gamma(\mathrm{z}) \right), \quad {}_{12}\psi_\gamma(\mathrm{z}) = \mathrm{i}\left({}_{jj}\psi_\gamma(\mathrm{z}) \right), \tag{57}$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit and ${}_{jj}\varphi_\gamma(\mathrm{z})$ are the potential functions given in (35)-(36) (Pobedrya, 1984). The unknown coefficients a_p that appear now in the relation (57) can be obtained as solution of following system of equations

$$a_p + B_1^1 V_2 \left(\chi^{(1)} - \frac{5S_4}{\pi^2} \right) a_1 \delta_{1p} + $$
$$- \sum_{k=1}^{\infty}{}^{*} \left[B_p^1 G_{kp} + B_p^1 C_k^0 r_{kp} + B_p^0 \left(A_p^0 \right)^{-1} \eta_{k\,p+2} - B_p^1 D_p^0 \eta_{k+2\,p} \right] a_k = \frac{1}{A_p^0} \left\| m \right\| R \delta_{1p}.$$

Solving the above system (Rodriguez-Ramos 2001), we have

$$a_1 = \frac{-E \left\| m \right\| R}{1 + H^- - \mathcal{V}_1 \mathcal{V}^{-1} \mathcal{V}_2}\,, \tag{58}$$

where

$$H^- = B_1^1 V_2 \left(\chi^{(1)} - 5S_4 / \pi^2 \right) - \left[B_1^1 G_{11} + B_1^1 C_1^0 r_{11} + B_1^0 \left(A_1^0 \right)^{-1} \eta_{13} - B_1^1 D_1^0 \eta_{31} \right],$$

$$\mathcal{V}_1 \left[v_{1s} \right] = \left[B_{4s+1}^1 G_{14s+1} + B_{4s+1}^1 C_1^0 r_{14s+1} + B_{4s+1}^0 \left(A_{4s+1}^0 \right)^{-1} \eta_{1\,4s+3} - B_{4s+1}^1 D_{4s+1}^0 \eta_{3\,4s+1} \right],$$
$$\mathcal{V}_2 \left(v_{12} \right) = \left[B_1^1 G_{4t+11} + B_1^1 C_{4t+1}^0 r_{4t+11} + B_1^0 \left(A_1^0 \right)^{-1} \eta_{4t+1\,3} - B_1^1 D_1^0 \eta_{4t+31} \right],$$
$$\mathcal{V} \left[v_{ts} \right] = \left[\delta_{ts} - B_{4s+1}^1 G_{4t+14s+1} - B_{4s+1}^1 C_{4t+1}^0 r_{4t+14s+1} - B_{4s+1}^0 \left(A_{4s+1}^0 \right)^{-1} \eta_{4t+1\,4s+3} + B_{4s+1}^1 D_{4s+1}^0 \eta_{4t+3\,4s+1} \right].$$

The effective coefficient C_{1212}^* is obtained from (21). Again it is necessary to apply the Green's theorem on the interface Γ and taking into consideration the expression (58), we can write the following expression for the shear module,

$$C_{1212}^* = C_{1212}^{(1)} + \left\| C_{1212} \right\| V_2 M\,, \tag{59}$$

where $M = \dfrac{(\kappa_1 + 1)E}{1 + H^{-} + \mathcal{V}_1 \mathcal{V}^{-1} \mathcal{V}_2}$.

The antiplane problem ${}_{13}\mathrm{L}$ and the effective coefficient C^{*}_{1313}

We have previously reported in López-Realpozo et al. (2011) the axial effective properties of composites with rhombic periodic cell. From formula (36) and (39) of this abovementioned work, the expression of C^{*}_{44} is computed as a particular case, i.e. when the basic cell is square. The final expression in this case is given by

$$C^{*}_{1313} = C^{*}_{2323} = C^{(1)}_{1313}\left(1 - V_2 \Lambda_3\right),$$

(60)where $\Lambda_3 = \dfrac{2\beta_1}{1 + \beta_1 V_2 - \beta_1 \aleph_1 \mathcal{Y}^{-1} \aleph_2}$, $\beta_p = \dfrac{(1-\kappa)K_s + \kappa p}{(1+\kappa)K_s + \kappa p}$, $\kappa = \mathrm{C}^{(2)}_{1313} / \mathrm{C}^{(1)}_{1313}$,

$\aleph_1(n_s) = \left(\beta_{4s-1}\, \eta_{1\,4s-1}\right)$, $\aleph_2(n_t) = \left(\eta_{4t-11}\right)$ and $\mathcal{Y}(n_{ts}) = \delta_{ts} - \beta^{2}_{4s-1} r_{4t-1\,4s-1}$.

Finally, as a summary, we have calculated the exact analytic formulae for the effective coefficients $\mathrm{C}^{*}_{1111}, \mathrm{C}^{*}_{1122}, \mathrm{C}^{*}_{1133}, \mathrm{C}^{*}_{3333}, \mathrm{C}^{*}_{1313}$ and C^{*}_{1212} given by the formulae (49)-(52), (59) and (60) for fibrous composites with square unit cell under spring imperfect contact at the interface. These closed–form expressions obtained show an explicit dependence on: (a) The mechanical properties of the constituents, (b) the volumetric fraction of each material phase and, (c) the dimensions of the representative volume of analysis and (d) the tangential imperfect interface parameters K_t and K_s.

Besides, the following connections between the effective properties for two phase elastic fibrous composites with transversely isotropic constituents and parallelogram periodic cell are obtained in a similar way as Guinovart-Díaz et al. (2001) and Rodriguez-Ramos et al. (2001),

$$\frac{\left(\mathrm{C}^{*}_{1111} + \mathrm{C}^{*}_{1122}\right)/2 - \left\langle\left(\mathrm{C}_{1111} + \mathrm{C}_{1122}\right)/2\right\rangle}{\mathrm{C}^{*}_{1133} - \left\langle \mathrm{C}_{1133}\right\rangle} = \frac{\mathrm{C}^{*}_{1133} - \left\langle \mathrm{C}_{1133}\right\rangle}{\mathrm{C}^{*}_{3333} - \left\langle \mathrm{C}_{3333}\right\rangle} = \frac{\left\|\left(\mathrm{C}_{1111} + \mathrm{C}_{1122}\right)/2\right\|}{\left\|\mathrm{C}_{1133}\right\|}. \tag{61}$$

These universal relations are the Hill's universal relations (see Hill 1964). Furthermore, (62) are valid for any shape of the interface Γ, they are independent of the fiber volume fraction and the imperfect parameters K_t.

Analysis of numerical results

In this section, some numerical results will be shown as a consequence of the obtained analytic expressions (49)-(52), (59) and (60). Moreover, comparisons with other theoretical results will be given in order to validate the aformentioned expressions.

The effect of interface in the overall properties of composites is studied in Fig. 2-4. The considered material constituents in these figures are isotropic with $C_{66}^{(2)} / C_{66}^{(1)} = 10$ and Poisson´s ratio taken as $\nu_1 = \nu_2 = 0.3$. Fig. 2 shows the ratio of the effective transverse shear moduli $C_{66}^{*} / C_{66}^{(1)}$ versus fiber volume fraction V_2 for different values of imperfect parameter K_t. Note that the imperfect parámeter K_t has a significant effect on C_{66}^{*}. For instance, K_t approches to infinity corresponds to perfect case bonding. Moreover, as K_t tends to zero the coefficient C_{66}^{*} asymptoticaly approaches to the solutions for pure sliding case. The effective shear modulus increases as the fiber volume fraction increases whereas this effective modulus decreases as K_t decreases as well. Fig. 3 displays the ratio of the bulk effective modulus k^{*}/k_1 versus fiber volume fraction V_2 for differents values of imperfect parameter K_t. This figure illustrates that k^{*} is independent of the imperfect parameter K_t, due to the fact of continuity of the displacement in the normal direction are considered, in contrast with the previous figure. Small differences are observed only for high-volume fraction. Similar behavior was reported by Jasiuk and Ton (1989).

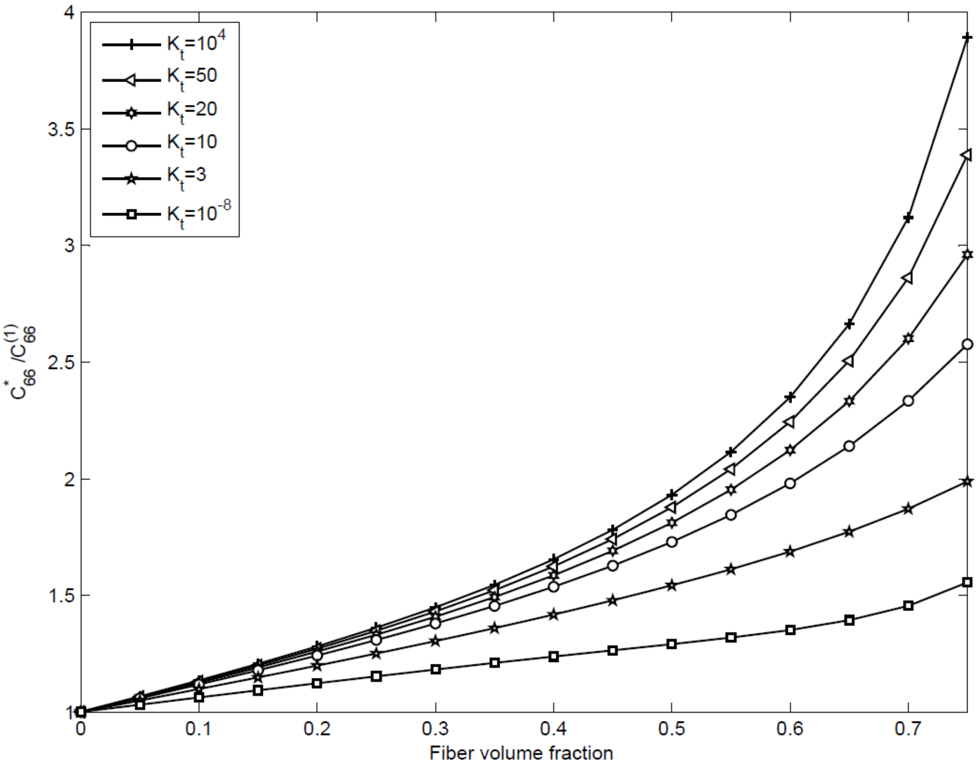

Fig. 2. Effect of imperfect parameter K_t on effective transverse shear modulus $C^*_{66} / C^{(1)}_{66}$ versus fiber volume fraction V_2.

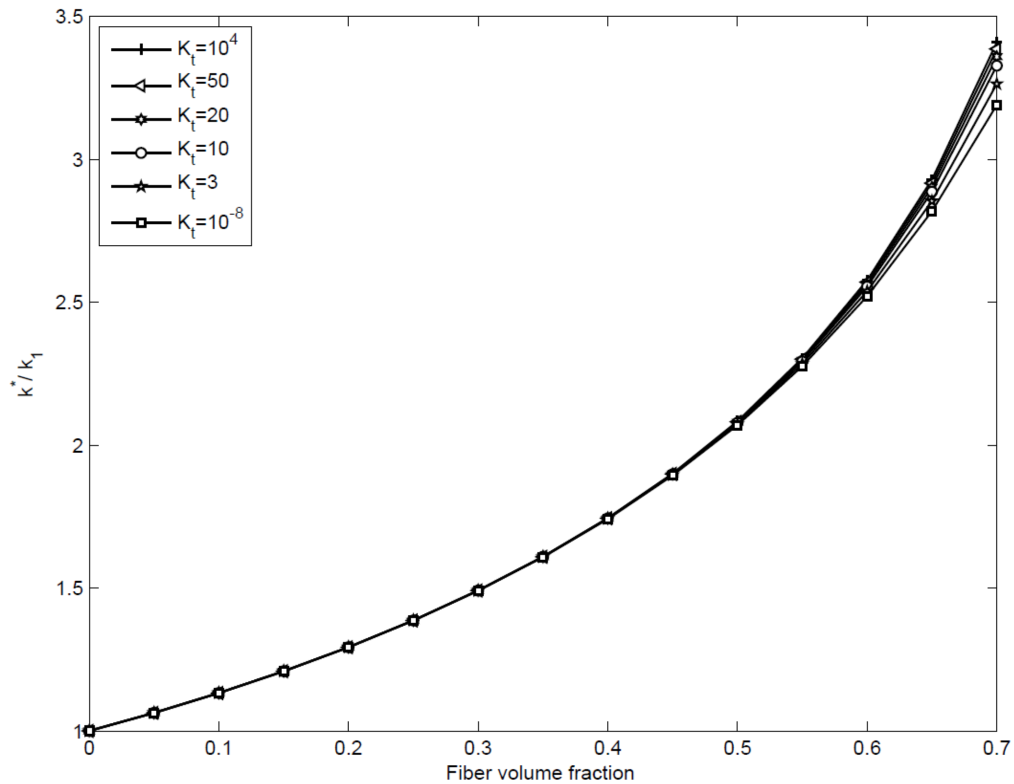

Fig. 3. The plane-strain bulk modulus for lateral dilatation without longitudinal extension $k^* = \left(C_{11}^* + C_{12}^*\right)/2$ versus fiber volume fraction V_2 for different values of imperfection parameter.

In Fig. 4 the change of the effective shear modulus $C_{66}^* / C_{66}^{(1)}$ with respect to the reciprocal imperfect parameter K_t for a fiber volume fraction $V_2 = 0.5$ is sketched. The present spring model is compared with two approaches, finite element method (FEM) reported in Otero et al. (2012) and three phase model published by Guinovart-Diaz et al. (2005). The local in-plane and out-plane problems in FEM are solved using the semi-analytical method combining asymptotic homogenization and finite element methods. Three phase model find analytically the solutions of the local problems, considering the existence of an isotropic thin layer interphase between the matrix and fibers. In this model, the properties of this intermediate layer are considered as a function of the imperfect parameters K_n, K_t and K_s, and the relation (19) reported by Hashin (2002) is used. In the present case $K_t = K_s = C_{66}^I \big/ V_I\, C_{66}^{(1)}$ and $K_n = 10^{12}$, where the capital superscript I denotes the intherphase property and the interphase volume fraction is

$V_I = 2.51 \cdot 10^{-4}$. Fig 4. exhibits good match between the spring, FEM and three phase models in all range of variation of $1/K_t$. Notice that C_{66}^* corresponds to the perfect case bonding for $K_t \to \infty$ whereas C_{66}^* approaches asymptoticaly to the solutions of fiber porous case as $K_t \to 0$.

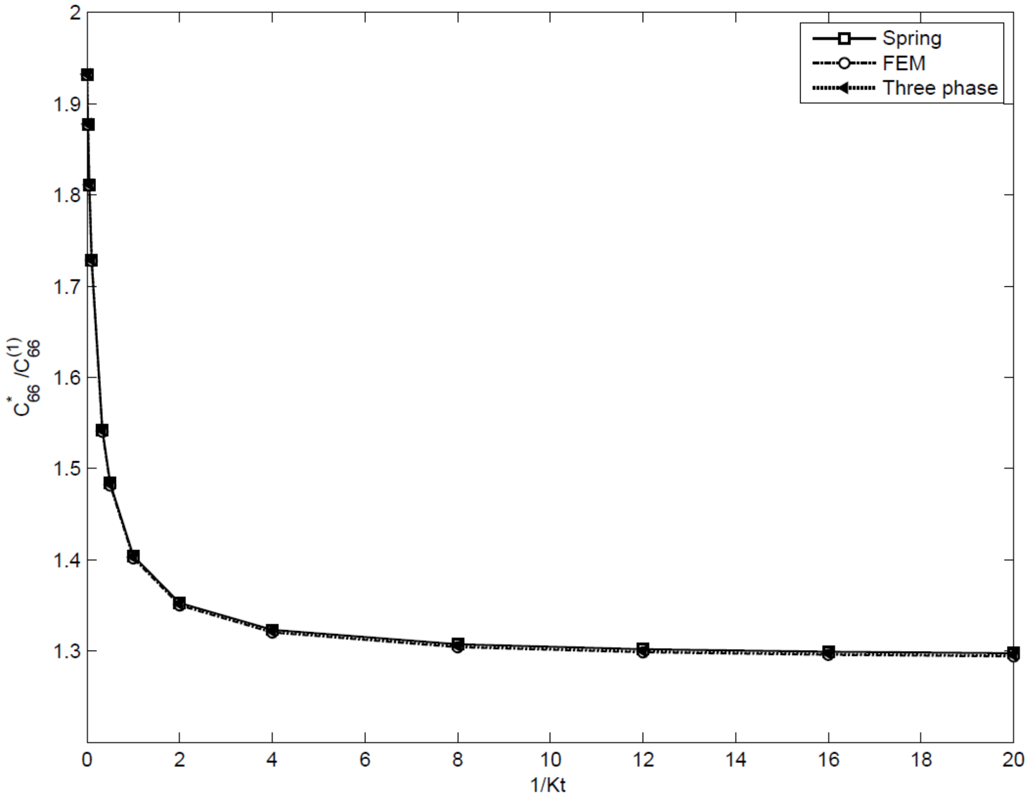

Fig. 4. The effective transverse shear moduli $C_{66}^* / C_{66}^{(1)}$ versus reciprocal imperfect parameter K_t for a fiber volume fraction $V_2 = 0.5$. The spring model is compared with finite element method (FEM) and three phase model.

All effective properties of fibrous composites with empty fibers are presented in Table 2. Young's modulus $E_1 = 70\,\text{GPa}$ and Poisson's ratio $\nu_1 = 0.3$ are the material parameters used in the computation. The set of all effective elastic coefficients are given for different values of porous volume fraction V_2. A comparison between the spring model and finite element method (FEM) Otero et al. (2012) is listed. The parameters K_n and K_t are zero for FEM and for the spring models is considered

$K_t = 0.01$ and the fibers properties are zero. The effective properties decrease as the porosity increase. It can be noticed a very good coincidence between these two models.

	V_2	0.05	0.2	0.35	0.55	0.75
C_{11}^*	FEM	80.525367	53.390014	36.536036	20.498584	5.849942
	Spring	80.525340	53.390011	36.536035	20.498584	5.849969
C_{12}^*	FEM	33.149807	18.365994	9.781059	3.378678	0.289122
	Spring	33.149803	18.365993	9.781059	3.378679	0.289123
C_{13}^*	FEM	34.102552	21.526802	13.895129	7.163179	1.841719
	Spring	34.102543	21.526801	13.895128	7.163179	1.841728
C_{33}^*	FEM	86.961532	68.916082	53.837077	35.797907	18.60504
	Spring	86.961526	68.916081	53.837077	35.797907	18.60504
C_{44}^*	FEM	24.358970	17.945057	12.915297	7.459089	2.111431
	Spring	24.358970	17.945057	12.915297	7.459089	2.111428
C_{66}^*	FEM	23.209442	13.422078	6.616327	1.808771	0.078901
	Spring	23.209425	13.422076	6.616326	1.808767	0.078818

Table 2: Comparison between the analytical results obtained by spring model and the finite element method (FEM) for porous fiber composites.

The following numerical calculations are conducted for three composites: glass/epoxy, carbon/epoxy and FP(Al_2O_3,)/Al reported in Hui-Zu and Tsu-Wei (1995). The carbon fiber is transversely isotropic, and all of the other constituents of the composites mentioned above are isotropic. The elastic properties of the fiber and matrix materials are listed in Table 3.

		Young´s modulus E (GPa)	Poisson's ratio ν	Shear modulus G (GPa)
Glass fiber		73.1	0.22	30
Epoxy resin		3.45	0.35	1.28
FP alumina fiber		379	0.2	157.9
Aluminium		68.9	0.345	25.6
Epoxy resin		5.35	0.354	1.98
Carbon fiber	Axial	232	0.279	5.03
	Transverse	15	0.490	--

Table 3. Elastic constants of constituents of composite glass/epoxy, carbon/epoxy and FP/Al.

To investigate the consistency of the calculation with experiments results, Fig. 5-6 show a comparison between AHM spring model for differents values of imperfect parameter K_t and experimental data reported by Tsai (1980) and Zhu et al. (1986) which can be indirectly taken from Figs.9 and 10 by Hui-Zu and Tsu-Wei (1995). Fig. 5 illustrates the analytical transverse Young's modulus E_t^* of the glass/epoxy composite. Figure 6 displays the analytical axial shear modulus G_a^* of the carbon/epoxy composite. In both figures, the corresponding experimental results are also plotted, showing good agreement between analytical and experimental results. Fig. 5 shows that experimental results constitute a dispersed cloud, which might be motivated by the effect of the contact between the phases. Observe that for different values of the imperfect parameter K_t the experiments are reached to the theoretical model. In Fig. 6 the experimental results are closed to the theoretical prediction for $K_t = 10^{10}$ which corresponds with perfect bonding. In Figs 5 and 6, lines with symbols "+" represent the analytical results assuming a composite with empty fiber reported by Sabina et al. (2002). The curves corresponding to the perfect bonding ($K_t = \infty \approx 10^{10}$) and empty fiber are upper and lower bounds respectively for all the curves obtained as the imperfect parameter K_t belongs to the interval $(0,\infty)$.

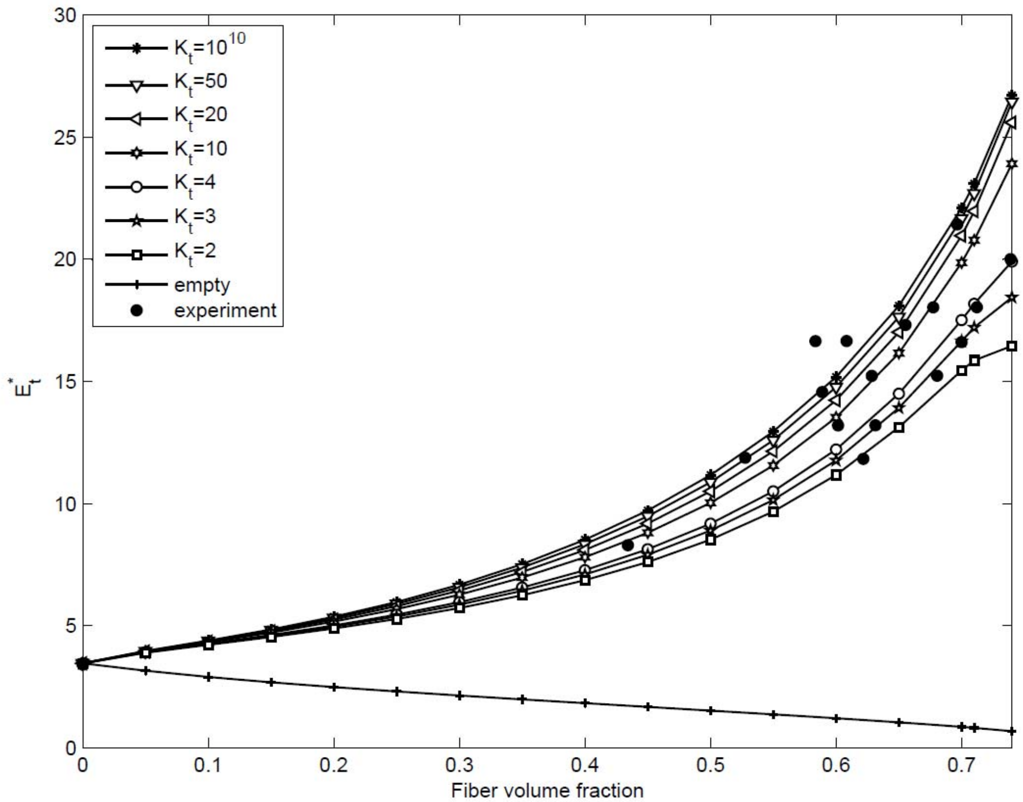

Fig. 5. The transverse Young's modulus E_t^* of glass/epoxy composite for various imperfect parameters are compared with experimental results. Perfect bonding (*) and empty fiber property (+) are upper and lower bound for different imperfect cases $0 < K_t < \infty$.

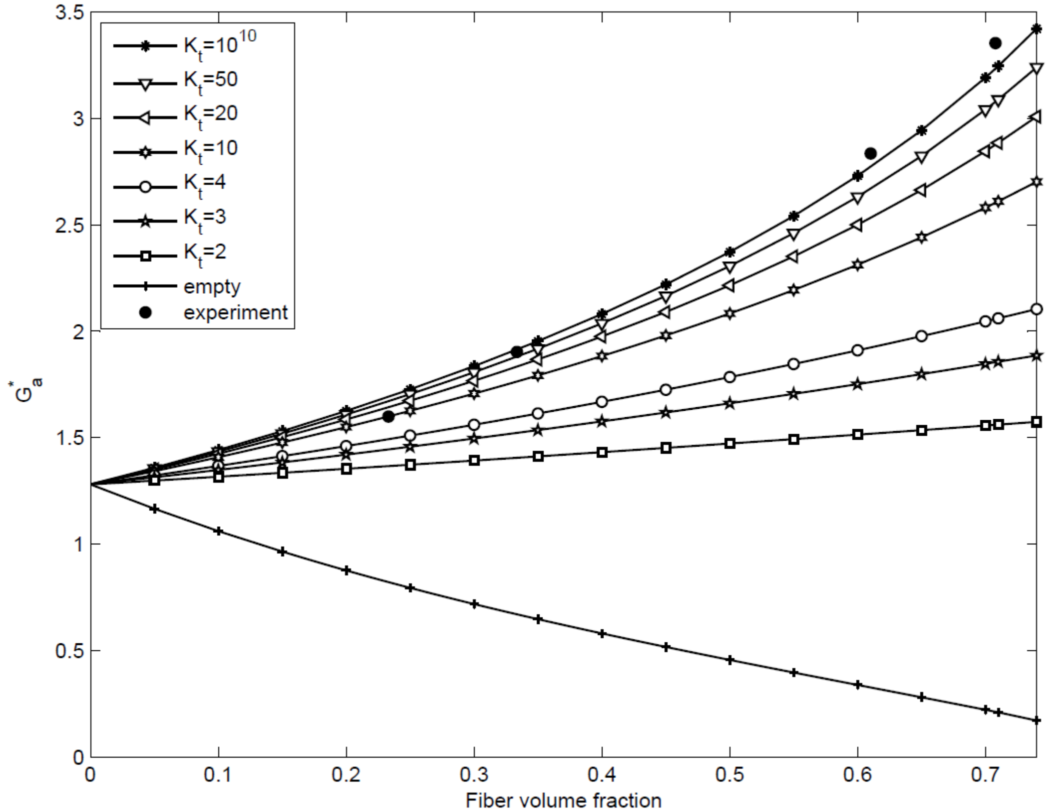

Fig. 6. The transverse shear axial modulus G_a^* of carbon/epoxy composite for various imperfect parameters are compared with experimental results.

Fig. 7 gives a comparison between the spring and three phase models for the six analytical engineering constants (axial and transverse Young , Poisson's ratio and shear moduli) for FP(Al_2O_3,)/Al transversely isotropic composite. In order to study the influence of the imperfect parameter, in the spring model, different values for the constant K_t are considered. The properties of the interphase are estimated assuming it as isotropic material in the three phase model. The symbols without color denote the spring model and with red color the three phase model. The coefficients are showing good agreement between the two analytical results. The lines with symbols "+" represent the analytical results assuming a composite with empty fiber.

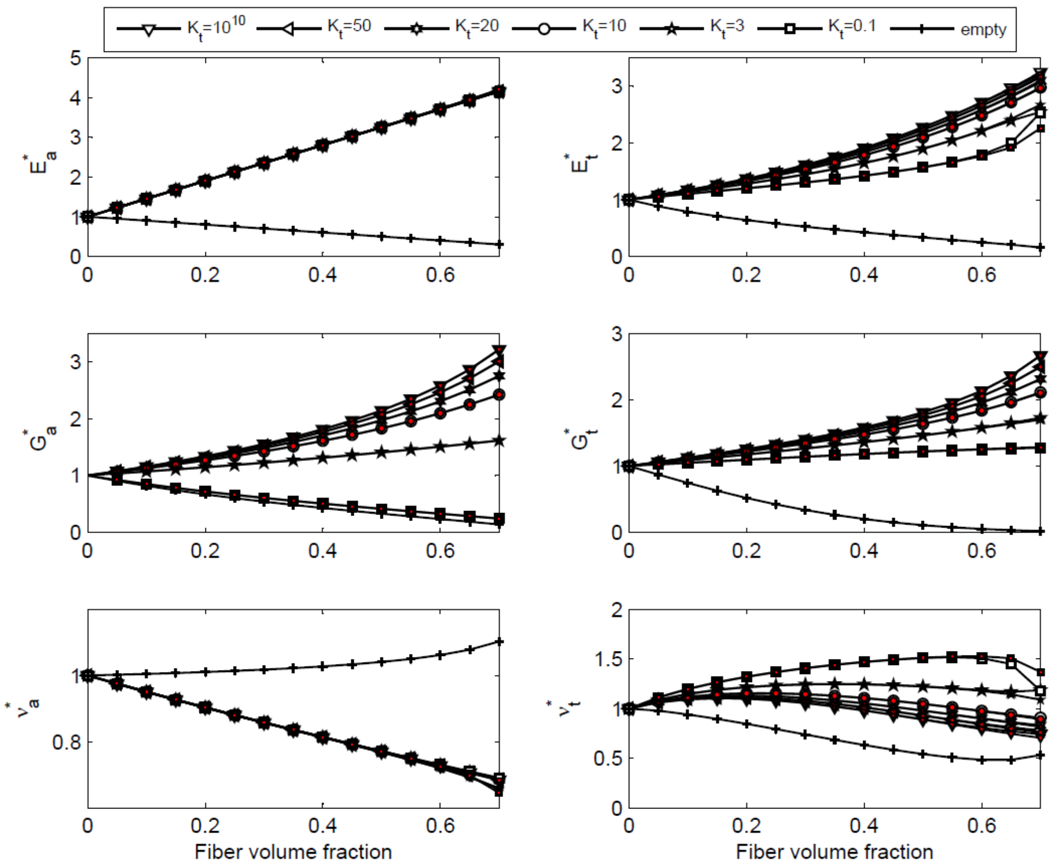

Fig. 7. Comparison between the spring and three phase models for FP(Al_2O_3,)/Al transversely isotropic composite. The axial and transverse Young , Poisson's ratio and shear moduli are calculate for different imperfect parameters. Empty fiber is denoted by the symbol "+".

Conclusions

In the paper analytical expressions are obtained using asymptotic homogenization method for all elastic coefficients considering imperfect contact conditions at the interface between the matrix and fiber. The formulae are functions of imperfect parameters.

The Hill universal relations are satisfied for these formulae. The following items are obtained from the previous numerical analysis where the interface effects are modeled as distributed mechanical springs.

(a) The mechanical behaviors of fibrous composite are influenced by the stiffness of the spring parameter. The present numerical results of the effective shear and Young

modulus versus the fiber volume ratio are correlated very well with other results for perfect and imperfect bonding case, and by the three phase model.

(b) At the fixed fiber volume ratio, the effective shear modulus of composite increases as the stiffness of the interphase increases.

(c) The analytical expressions are validated by experimental results.

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