

**Two approaches for the evaluation of the effective properties of elastic
composite with parallelogram periodic cells**

Reinaldo Rodríguez-Ramos¹, Harald Berger^{2,*}, Raúl Guinovart-Díaz¹,
Juan Carlos López-Realpozo¹, Mathias Wörkner²,
Ulrich Gabbert², Julián Bravo-Castillero¹

¹Facultad de Matemática y Computación, Universidad de La Habana, San Lázaro y L,
Vedado, Habana 4, CP-10400, Cuba

jclrealpozo@matcom.uh.cu, guino@matcom.uh.cu, reinaldo@matcom.uh.cu,
jbravo@matcom.uh.cu

²Institute of Mechanics, University of Magdeburg, Universitätsplatz 2,
D-39106 Magdeburg, Germany

mathias.wuerkner@st.ovgu.de, harald.berger@ovgu.de, ulrich.gabbert@ovgu.de

* corresponding author: harald.berger@ovgu.de

Abstract

In this work, a two-phase parallel fiber-reinforced periodic elastic composite is considered wherein the constituents exhibit transverse isotropy. Effective properties of fibrous composites by means of the asymptotic homogenization method (AHM) and numerical homogenization using finite element method (FEM) are calculated for different types of parallelogram cells. Some numerical examples and comparisons with other theoretical results demonstrate that both methods are efficient for the analysis of composites with presence of parallelogram cells. The effects of the configuration of the cells on the effective

properties are observed. In general, both models predict the monoclinic behavior of the composites.

Keywords: Effective properties, double periodic, fibrous composites, finite element, asymptotic homogenization

1. Introduction

The response of unidirectional composites with parallel fibers axes has been extensively characterized only for square and hexagonally packed fiber-matrix arrays. However, there exist some composites that retain rectangular packing features. In this sense, there are a number of numerical unit cell based results on the mechanical behavior periodic rectangular arrays of aligned fibers, especially from the “early days” of the use of Finite Elements in micromechanics, e.g., Adams (1970); Foye (1970); Lin et al. (1972), which, however, did not go into details of the anisotropic elastic behavior.

More recently, analytic expressions for elastic effective coefficients of fibrous composites with isotropic elastic constituents for square and hexagonal cells under perfect contact conditions at the interfaces are calculated in many previous works. In Rodríguez-Ramos et al. (2001), Guinovart-Díaz et al. (2001) and Molkov and Pobedria (1985) analytical expressions for the effective properties are obtained using asymptotic homogenization method (AHM). On the other hand, finite element method (FEM) is applied in Berger et al. (2005) in order to obtain effective properties of elastic composites. In Golovchan and Nikityuk (1981) a very attractive method is presented for the solution of the problem on the shear of a regular fibrous medium underlying, which is the exact solution of the Laplace equation in a strip with an infinite number of circular holes. Only single, quite convergent series are used here. Such approach permits obtaining values of the elastic moduli for

different angle of inclination of the cells. Recently, in Guinovart-Díaz et al. (2011) the effective elastic moduli of two-phase fibrous periodic composites are obtained for different types of parallelogram cells based on the AHM and making use of potential methods of a complex variable and properties of elliptic and related functions. The constituents exhibit transversely isotropic properties. A doubly period-parallelogram array of cylindrical inclusions under longitudinal shear is considered. The behavior of the shear effective coefficient for different geometry arrays of the cell related to the angle of the fibers is studied.

In this work micromechanical analysis methods are applied to unidirectional fiber composites with different cross angles of the cells to determine the homogenized elastic properties of the composite. In particular, all elastic coefficients are calculated by AHM and FEM for different geometry arrays of the cell in two-phase fibrous periodic composites. This work exhibits the progress of the recently published papers in this journal (Guinovart-Díaz et al., 2011 and Würkner et al., 2011). Since the theoretical fundamentals of both methods are developed in detail separately in previous works, a comparison between the results of the elastic effective coefficients derived for both methods is convenient and it is presented here. Moreover, the difference of the present work with respect to other previous works consists in the computation of all effective coefficients for different unidirectional parallelogram fiber distributions where the symmetry lines define a parallelepiped unit cell, representing the periodic microstructure of this fibrous composite. The effects of the geometrical distributions and the angle of the fibers in the composite are analyzed. The results of this paper are mainly based on the impact of the fiber cross angles on the stiffness properties obtaining a composite with monoclinic structure characterized by 13

homogenized elastic coefficients (Royer and Dieulesaint, 2000). In particular, for rhombic periodic cells with 60° and 90° composites with hexagonal and tetragonal crystal classes are obtained (Rodríguez-Ramos et al., 2001, Guinovart-Díaz et al., 2001), respectively. The accuracy of the results from both micromechanics modeling procedures has been compared.

2. Asymptotic homogenization method

We consider elastic materials, that respond linearly to changes in the mechanical stress and strain tensors. A two-phase uniaxial reinforced material is considered here in which fibers and matrix have transversely isotropic elastic properties; the axis of transverse symmetry coincides with the fiber direction, which is taken as the Ox_3 axis. The fiber cross-section is circular. Moreover, the fibers are periodically distributed without overlapping in directions parallel to the Ow_1 -and Ow_2 -axis, where $w_1 \neq 0$ and $w_2 \neq 0$ ($w_2 \neq \lambda w_1$, $\lambda \in \mathbb{R}$) are two complex numbers which define the parallelogram periodic cell of the two-phase composite. The composite consists of a parallelogram array of identical circular cylinders embedded in a homogeneous medium. The cylinders are infinitely long. As shown in Fig. 1, the infinitely extended doubly-periodic structure is obtained from a primitive cell which is repeated in the two directions, where w_1 and w_2 denote the two fundamental periods. The general period P_{st} can be defined as $P_{st} = s w_1 + t w_2$, where s and t are arbitrary integers.

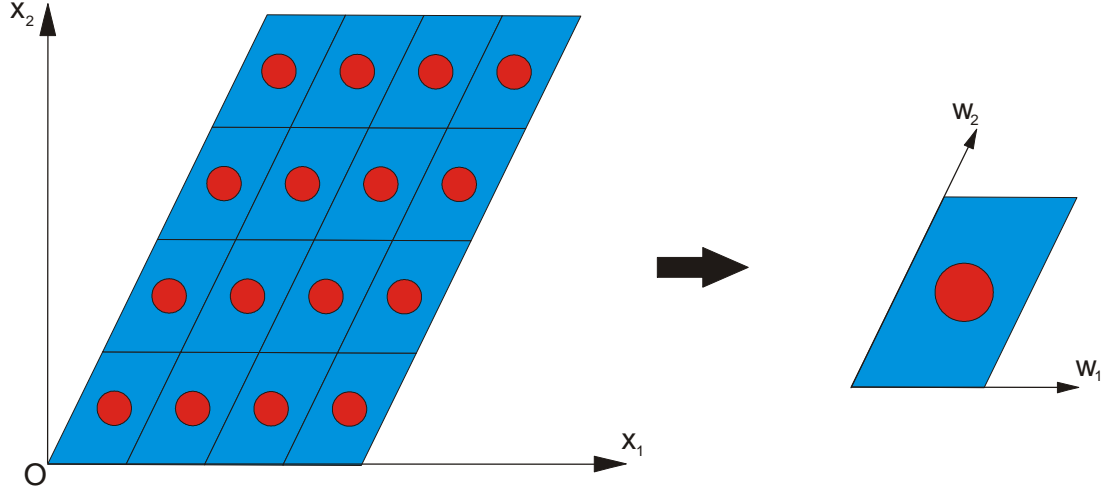


Fig. 1. The heterogeneous medium and extracted parallelogram periodic cell

As the body forces are absent, stress σ , strain ϵ and displacement u fields satisfy the following equations, respectively

$$\sigma_{ij,j} = 0, \quad (2.1)$$

$$\sigma_{ij} = C_{ijkl} \epsilon_{kl}, \quad \epsilon_{kl} = \frac{1}{2} \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right). \quad (2.2)$$

Substituting (2.2) into (2.1) we obtain a system of partial differential equations with coefficients rapidly oscillating

$$\left(C_{ijkl}(\mathbf{y}) u_{l,k} \right)_{,j} = 0, \quad \text{on } \Omega. \quad (2.3)$$

The overall properties of the above periodic medium are sought by means of the well-known asymptotic homogenization method (AHM) (Rodríguez-Ramos et al., 2001, Guinovart-Díaz et al., 2001). Then, it follows that in terms of the fast variable \mathbf{y} , the appropriate periodic unit cell Y is taken as a regular parallelogram in the $y_1 y_2$ -plane so that $Y = Y_1 \cup Y_2$ with $Y_1 \cap Y_2 = \emptyset$, where the domain Y_1 is occupied by the matrix and its

complement Y_2 , a circle of radius R , is filled up with the fiber (Fig. 2). The common interface between the fiber and the matrix is denoted by Γ . Beside the use of subscript, matrix and fiber associated quantities are also referred below by means of superscripts in brackets (1) and (2), respectively.

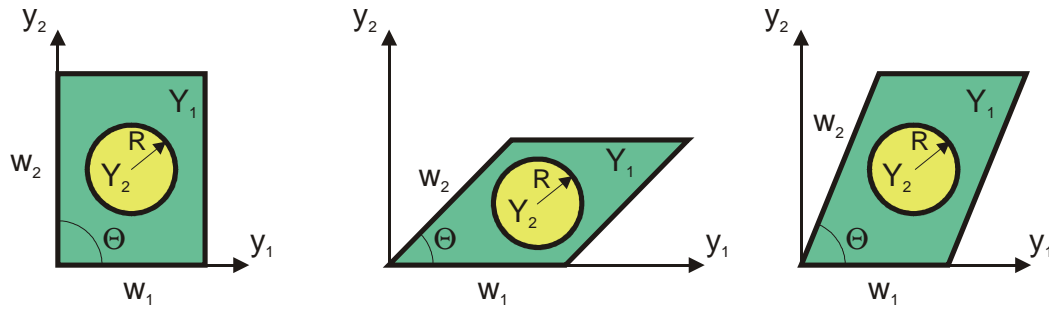


Fig. 2. Different unit cells - rectangle, rhombic and parallelogram

The constitutive relations of the linear elasticity theory for a heterogeneous and periodic medium, Ω , is characterized by the Y -periodic function $\mathbf{C}(\mathbf{y})$ where \mathbf{C} is the elastic fourth order tensor. By means of AHM, the original constitutive relations with rapidly oscillating material coefficients are transformed in new physical relations with constant coefficients \mathbf{C}^* , which represent the elastic properties of an equivalent homogeneous medium and are called the effective coefficients of Ω . Therefore, the system (2.3) can be transformed into an equivalent system with constant coefficients which represent the overall properties of the composite.

The system of equations (2.3) under suitable boundary conditions can be solved asymptotically posing the ansatz

$$\mathbf{u}(\mathbf{x}) = \tilde{\mathbf{u}}(\mathbf{x}) + \varepsilon \tilde{\mathbf{u}}_1(\mathbf{x}, \mathbf{y}) + O(\varepsilon^2),$$

where $\tilde{\mathbf{u}}$ satisfy the homogenized system of differential equations

$$C_{ijkl}^* \tilde{u}_{l,kj} = 0, \quad \text{on } \Omega,$$

and the superscript '*' denotes the overall property. The term $\tilde{\mathbf{u}}_1$ represent a correction of the $\tilde{\mathbf{u}}$ respectively. The function $\tilde{\mathbf{u}}_1$ is found in combination of the function $\mathbf{U}(\mathbf{y})$ (solution of the local problems) and the partial derivatives of the function $\tilde{\mathbf{u}}$.

The main problem to obtain such average formulae is to find the Y – periodic solutions $U_{k(pq)}$ of the six L_{pq} elastic local problems on Y in terms of the fast variable \mathbf{y} (Rodríguez-Ramos et al., 2001, Guinovart-Díaz et al., 2001). Once the local problems are solved, the homogenized moduli C_{ijpq}^* may be determined by using the following formulae

$$C_{ijpq}^* = \left\langle C_{ijpq} + C_{ijkl} U_{k(pq),l} \right\rangle, \quad (2.4)$$

$$\text{where } \langle \bullet \rangle = \frac{1}{|Y|} \int_Y \bullet dY.$$

Due to the periodic distributions of the fibers in the plane Ow_1w_2 it is possible to reduce the general problem to the solution of the local problems over the periodic cell. In this case, the elasticity tensor components C_{ijkl} take different values in the regions occupied by these two different materials, such that the local problems can be written as

$$\tau_{ij(pq),j}^{(\alpha)} = 0 \quad \text{if } (y_1, y_2) \in Y_\alpha \quad (2.5)$$

with $\tau_{ij(pq)}^{(\alpha)} = C_{ijkl}^{(\alpha)} U_{k(pq),l}^{(\alpha)}$ ($\alpha, l, j = 1, 2$ and $i, k, p, q = 1, 2, 3$), and the phases of the composite are denoted by Y_1 -matrix and Y_2 -fiber. Here the comma notation denotes a partial

derivative relative to the y_j component; the summation convention is understood, but it is taken only over repeated lower case indices; in general, the range for Latin and Greek, upper and lower case, indices runs from 1 to 3 and from 1 to 2, respectively.

The solution of the equation (2.5) must consist of doubly periodic functions in y_1 and y_2 subjected to the following perfect bonding conditions (continuity of displacements and stresses) at the interface Γ and periodic conditions

$$U_{k(pq)}^{(1)} = U_{k(pq)}^{(2)} \quad \text{on } \Gamma, \quad (2.6)$$

$$(\tau_{ij(pq)}^{(1)} + C_{ijpq}^{(1)})n_j^{(1)} = -(\tau_{ij(pq)}^{(2)} + C_{ijpq}^{(2)})n_j^{(2)} \quad \text{on } \Gamma \quad (2.7)$$

$$\langle U_{i(pq)} \rangle = 0, \quad (2.8)$$

where the outward unit normal vector to the interface Γ is denoted by \mathbf{n} .

The potential method of complex variables $z = y_1 + iy_2$, $(y_1, y_2) \in Y$ and the properties of

doubly periodic Weierstrass $\wp(z) = \frac{1}{z^2} + \sum_{s,t} \left\{ \frac{1}{(z - P_{st})^2} - \frac{1}{P_{st}^2} \right\}$ and related functions (Z-

function $\zeta(z) = -\wp'(z)$ and Natanzon's function $Q(z)$) are used for the solution of the

local problems (2.5)-(2.8). Hence, the non-zero solution $U_{k(pq)}^{(\alpha)}$ in Y_α of the problem (2.5)-

(2.8) must be found among doubly periodic functions of periods w_1, w_2 (see Fig. 1). Each

local problem (2.5)-(2.8) uncouples into sets of equations. An in-plane strain system for

$U_{\lambda(pq)}^{(\alpha)}$, $\lambda = 1, 2$ and an out-of-plane strain Laplace's equation for $U_{3(pq)}^{(\alpha)}$ has to be solved.

Then the solution of the in-plane (out-of-plane) strain problems involves the determination

of the in-plane (out-of-plane) displacements, strains and stresses over each phase Y_α of the

composite. Due to the non-vanishing components of the elastic tensors $C_{ijpq}^{(\alpha)}$, the only non-

homogeneous problems, that have a non-zero solution, correspond to the four in-plane strain problems L_{pp} and L_{12} , and the two out-of-plane strain ones L_{23} and L_{13} . In that way the solutions of both (in-plane and out-of-plane) local problems (Guinovart-Díaz et al., 2001, Rodriguez-Ramos et al., 2001) (six L_{pq} problems need to be considered due to the symmetry between p and q) lead to obtain the average coefficients of the composite given in Fig. 1.

From the solutions of the local in-plane problems L_{pp} and L_{12} the coefficients $C_{1111}^*, C_{1122}^*, C_{1133}^*, C_{2222}^*, C_{2233}^*, C_{3333}^*, C_{1112}^*, C_{2212}^*, C_{3312}^*, C_{1212}^*$ are derived, whereas the shear coefficients $C_{2323}^*, C_{2313}^*, C_{1313}^*$ (Guinovart-Díaz et al., 2011, Rodriguez-Ramos et al., 2011) are obtained from out-of-plane problems L_{23} and L_{13} . The closed form formulae of the effective coefficients can be listed as follows

$$\begin{aligned}
C_{1111}^* &= \langle C_{1111} \rangle + V_2 (\|m\| - \|k\| \Re\{K_{11}^+\}), & C_{1112}^* &= V_2 \|m\| (\chi_1 + 1) \Im\{\bar{a}_{1(11)}\}, \\
C_{1122}^* &= \langle C_{1122} \rangle - V_2 (\|m\| + \|k\| \Re\{K_{22}^+\}), & C_{2212}^* &= V_2 \|m\| (\chi_1 + 1) \Im\{\bar{a}_{1(22)}\}, \\
C_{1133}^* &= \langle C_{1133} \rangle - V_2 \|C_{1133}\| \Re\{K_{33}^+\}, & C_{3312}^* &= V_2 \|C_{1133}\| (\chi_1 + 1) \Im\{\bar{a}_{1(33)}\}, \\
C_{2233}^* &= \langle C_{2233} \rangle - V_2 \|C_{1133}\| \Re\{K_{33}^-\}, & C_{1313}^* &= C_{1313}^{(1)} - 2V_2 \Re\{\bar{a}_{1(13)}\}, \\
C_{2222}^* &= \langle C_{2222} \rangle + V_2 (\|m\| - \|k\| \Re\{K_{22}^-\}), & C_{2313}^* &= 2C_{2323}^{(1)} V_2 \Im\{\bar{a}_{1(13)}\}, \\
C_{3333}^* &= \langle C_{3333} \rangle - V_2 \|C_{1133}\|^2 \Re\left\{\frac{\chi_2 \Delta_3 - \bar{\Delta}_3}{m_1}\right\}, & C_{2323}^* &= C_{2323}^{(1)} + 2V_2 \Im\{\bar{a}_{1(23)}\}. \\
C_{1212}^* &= C_{1212}^{(1)} - \|m\| (\chi_1 + 1) V_2 \Im\{\bar{a}_{1(12)}\}, & &
\end{aligned} \tag{2.9}$$

where $K_{pp}^\pm = \frac{\|k\|}{m_1} [\chi_2 \Delta_p - \bar{\Delta}_p] \pm (\chi_1 + 1) \bar{a}_{1(pp)}$ and the over bar denotes complex conjugate

numbers. The magnitudes χ_α , $\alpha = 1, 2$ and Δ_p are given in the Appendix.

This derivation leads to thirteen analytic expressions for the effective coefficients of the composite, which depend on the properties of each component, the periods of the parallelogram and the volume fraction of the fiber $V_2 = \frac{\pi R^2}{V}$, whereas $k = \frac{C_{1111} + C_{1122}}{2}$,

$m = \frac{C_{1111} - C_{1122}}{2}$ and $a_{1(pq)}$ is the solution of the infinite system of algebraic equations for

each pq local problem. In particular, the expressions for $a_{1(13)}$, $a_{1(23)}$ are reported in

Guinovart-Díaz et al. (2011) and Rodríguez-Ramos et al. (2011). The unknown constants

$a_{k(pp)}$ related to the plane local problems L_{pp} are the solutions of the system

$$(I + H_1 \delta_{1k} + W)X + (H_2 \delta_{1k} + M)\bar{X} = H_3, \quad (2.10)$$

where $X(a_{k(pp)})$ is the unknown vector, I is the unit matrix, δ_{1k} is the Kronecker's delta

function and the numeric matrices W , M and the column vectors H_1, H_2, H_3 are given by

the expressions reported in the Appendix. For the other local plane problem L_{12} , the system

and the above magnitudes are the same with the only different expression $H_3(h_n) = i E \delta_{1n}$.

Besides, the following connections between the effective properties for two phase elastic

fibrous composites with transversely isotropic constituents and parallelogram periodic cell

are obtained in a similar way as Guinovart-Díaz et al. (2001) and Rodríguez-Ramos et al.

(2001),

$$\frac{C_{1111}^* + 2C_{1122}^* + C_{2222}^* - 2\langle C_{1111} + C_{1122} \rangle}{C_{1133}^* + C_{2233}^* - \langle C_{1133} \rangle} = \frac{C_{1133}^* + C_{2233}^* - \langle C_{1133} \rangle}{C_{3333}^* - \langle C_{3333} \rangle} = \frac{\|C_{1111} + C_{1122}\|}{\|C_{1133}\|}, \quad (2.11)$$

$$\frac{C_{1112}^* + C_{2212}^*}{C_{3312}^*} = \frac{\|C_{1111} + C_{1122}\|}{\|C_{1133}\|}. \quad (2.12)$$

These universal relations seem to be new. In particular, for hexagonal and square arrays the expression (2.11) are the Hill's universal relations (see Hill 1964). Furthermore, (2.11)-(2.12) are valid for any shape of the interface Γ , they are independent of the fiber volume fraction and the periodic cells (rectangular, rhombic and parallelogram).

3. Numerical homogenization

The numerical homogenization procedure is based on finite element analysis. Therefore appropriate finite element models for the unit cells must be developed, which ensure the periodicity and give the possibility to calculate the full set of effective coefficients.

The periodicity condition stated in Suquet (1987) can be defined in the following form

$$u_i^{A_k^+} - u_i^{A_k^-} = \epsilon_{ij}^0 (y_j^{A_k^+} - y_j^{A_k^-}). \quad (3.1)$$

The values $u_i^{A_k^+}$ and $u_i^{A_k^-}$ are the i -th displacement components on the boundary surfaces A_k^- and A_k^+ of the unit cell, which are perpendicular to the y_k -axis ('+' for outward normal direction identical to positive axis direction, '-' for outward normal direction identical to negative axis direction). ϵ_{ij}^0 are in general given constants. In order to calculate all stiffness coefficients C_{ijkl}^* six special load cases must be created, which produce six strain states where only one strain component is non-zero in every case. Then from constitutive relation

$$\langle \sigma_{ij} \rangle = C_{ijkl}^* \langle \epsilon_{kl} \rangle, \quad (3.2)$$

because only one component $\langle \epsilon_{kl} \rangle$ is non-zero, we can get separate equations for the calculation of all coefficients

$$C_{ijkl}^* = \frac{\langle \sigma_{ij} \rangle}{\langle \epsilon_{kl} \rangle} \quad \begin{matrix} i, j, k, l = 1, 2, 3 \\ k = l \end{matrix}, \quad C_{ijkl}^* = \frac{\langle \sigma_{ij} \rangle}{\langle 2\epsilon_{kl} \rangle} \quad \begin{matrix} i, j, k, l = 1, 2, 3 \\ k \neq l \end{matrix}. \quad (3.3)$$

E.g. in case that $\langle \epsilon_{11} \rangle$ is non-zero, we get $C_{1111}^* = \langle \sigma_{11} \rangle / \langle \epsilon_{11} \rangle$, $C_{2211}^* = \langle \sigma_{22} \rangle / \langle \epsilon_{11} \rangle$, etc.

In Eqs. (3.2) and (3.3) the quantities $\langle \sigma_{ij} \rangle$ and $\langle \epsilon_{kl} \rangle$ are averaged values with respect to the unit cell Y

$$\langle \sigma_{ij} \rangle = \frac{1}{|Y|} \int_Y \sigma_{ij}(y) dy, \quad (3.4)$$

$$\langle \epsilon_{kl} \rangle = \frac{1}{|Y|} \int_Y \epsilon_{kl}(y) dy. \quad (3.5)$$

Because of finite element discretization the integrals in Eqs. (3.4) and (3.5) are changed to

$$\langle \sigma_{ij} \rangle = \frac{1}{|Y|} \sum_e \sigma_{ij}^e |Y_e|, \quad (3.6)$$

$$\langle \epsilon_{kl} \rangle = \frac{1}{|Y|} \sum_e \epsilon_{kl}^e |Y_e|. \quad (3.7)$$

The quantities σ_{ij}^e , ϵ_{kl}^e are averaged element values for the corresponding coefficients of the stress and strain tensor and $|Y_e|$ is the volume of a finite element.

The six necessary load cases can be produced by prescribed displacement differences between opposite surfaces. For producing pure normal strains we apply

load case 1: non-zero strain $\langle \epsilon_{11} \rangle$: $u_1^{A_1^+} - u_1^{A_1^-} = \bar{u}$,

load case 2: non-zero strain $\langle \epsilon_{22} \rangle$: $u_2^{A_2^+} - u_2^{A_2^-} = \bar{u}$,

load case 3: non-zero strain $\langle \epsilon_{33} \rangle$: $u_3^{A_3^+} - u_3^{A_3^-} = \bar{u}$

and for pure shear strain

load case 4: non-zero strain $\langle \epsilon_{23} \rangle$: $u_2^{A_3^+} - u_2^{A_3^-} = \bar{u}$,

load case 5: non-zero strain $\langle \epsilon_{13} \rangle$: $u_1^{A_3^+} - u_1^{A_3^-} = \bar{u}$,

load case 6: non-zero strain $\langle \epsilon_{12} \rangle$: $u_1^{A_2^+} - u_1^{A_2^-} = \bar{u}$.

Here \bar{u} is an arbitrary non-zero value. For simplicity $\bar{u}=1$ is chosen. To ensure the full periodicity in every load case all remaining displacement differences are set to zero. To apply these displacement differences opposite nodal pairs are coupled by appropriate constraint equations. For that a special meshing procedure ensures identical mesh configurations on opposite surfaces. To avoid rigid body movement one arbitrary node must be fixed in all directions. We used the corner node at origin of coordinate system.

In case of non rectangular cells, like rhombic and parallelogram shape, load case one produces also a shear strain $\langle \epsilon_{12} \rangle$ due to applying the displacements in y_1 direction on oblique surfaces. To overcome this lack additional displacement differences must be applied to compensate this unwanted shear strain. Therefore load case one must be modified in the following form

load case 1 (for oblique cells): non-zero strain $\langle \epsilon_{11} \rangle$: $u_1^{A_1^+} - u_1^{A_1^-} = \bar{u}$,

$$u_1^{A_2^+} - u_1^{A_2^-} = \bar{u} \cdot |w_2| \cos(\theta) .$$

For finite element analysis software package ANSYS is used. Here algorithms are written in APDL (ANSYS Parametric Design Language), which is delivered by the software and it makes the handling much more comfortable. The models of the unit cells (see Fig. 3) consist of three-dimensional SOLID226-elements. These elements are characterized by twenty nodes and quadratic shape functions. In y_3 direction one element is sufficient.

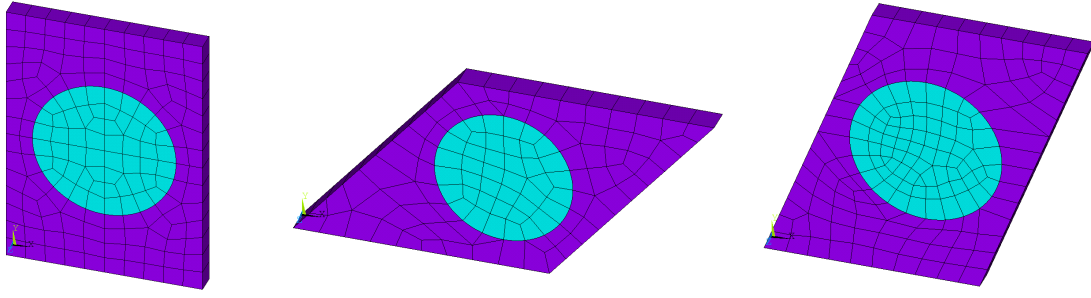


Fig. 3. Meshed RVEs for rectangle, rhombic and parallelogram cell

4. Analysis of results

The thirteen effective properties of the composite were calculated using AHM (analytical formulae (2.9)) and FEM (numerical formulae (3.3)) involving the volume fraction of the constituents and their properties as well as the configuration of the fibers in the composites. The effective properties by both methods strongly depend on the infinitely extended doubly-periodic structure obtained from a parallelogram periodic cell. The composite with rhombic cell and $\theta = 60^\circ$ or $\theta = 90^\circ$ (Fig. 1) exhibits hexagonal or tetragonal symmetric classes, respectively, and as a particular case five or six overall properties are obtained (Rodríguez-Ramos et al., 2001, Guinovart-Díaz et al., 2001). Moreover, the behavior of composites with parallelogram periodic cells shows monoclinic symmetry.

The finite element predictions (FEM) of the elastic moduli are compared with analytical solutions by AHM and the validation of both methods are given in Tables 1-7. For simplicity, the effective properties are written in the short index notation. The computations by AHM were made for $N_0 = 10$, where N_0 denotes the number of equations considered in the solution of the infinite algebraic system of equation (2.10). In general, for low volume fraction of fiber ($V_2 < 0.4$) the accuracy and convergence of the results are good

for small value of N_0 ($N_0 \leq 2$). Only for a high volume fraction the value of N_0 requires to be changed for getting better accuracy. The shear ratio is fixed, $G^{(2)}/G^{(1)}=120$ in all calculations (except in one case in Table 4). Young's moduli and Poisson's ratios are artificial values, in particular, the following material properties were chosen: matrix $E^{(1)} = 2.6$ GPa, $\nu^{(1)} = 0.3$, fiber $E^{(2)} = 312$ GPa, $\nu^{(2)} = 0.3$.

The results obtained by AHM and FEM coincide in a wide variety of cases. Very good match between both approaches can be seen in Tables 1-6. In Table 1 different elastic composites with rectangular periodic cell ($w_1 = 1$, $w_2 = ci$, $c = 1.1, 1.25, 1.5$) are considered. Average properties for in-plane and out-of-plane properties are shown vs. fiber volume fraction. The global behavior of the composite is orthotropic and it depends on the configuration of the rectangular cell. Eight of them are shown in this Table 1. The principal discrepancies are concerned only for transversal effective properties of composites with high volume fraction and large distance between the fibers ($c=1.5$).

Table 1: Effective properties by AHM and FEM for a composite with rectangular periodic cell for different ratios $|w_2|/|w_1|$ and fiber volume fractions

$ w_2 / w_1 $	V_2	C_{11}^*		C_{12}^*		C_{13}^*		C_{23}^*	
		AHM	FEM	AHM	FEM	AHM	FEM	AHM	FEM
1.1	0.1	4.0757	4.0770	1.6972	1.6965	1.7319	1.7321	1.7288	1.7289
	0.3	5.9477	5.9496	2.1087	2.1093	2.4169	2.4177	2.3596	2.3610
	0.5	10.347	10.320	2.4968	2.4945	3.8533	3.8443	3.4096	3.4158
1.25	0.1	4.0847	4.0849	1.6957	1.6937	1.7341	1.7336	1.7268	1.7271
	0.3	6.1334	6.1371	2.0892	2.0897	2.4668	2.4680	2.3256	2.3274
	0.5	12.207	12.286	2.3617	2.3904	4.3705	4.4029	3.2052	3.2177
1.5	0.1	4.1016	4.1033	1.6917	1.6929	1.7380	1.7389	1.7236	1.7243
	0.3	6.5184	6.5473	2.0439	2.0308	2.5687	2.5734	2.2780	2.2761
	0.5	20.791	24.281	2.3630	2.3475	6.6674	7.9885	2.8029	3.0248

		C_{33}^*		C_{44}^*		C_{55}^*		C_{66}^*	
1.1	0.1	34.578	34.571	1.2152	1.2158	1.2212	1.2218	1.1567	1.1575
	0.3	96.853	96.851	1.7984	1.7992	1.8864	1.8879	1.5287	1.5294
	0.5	159.48	159.48	2.7861	2.7836	3.3100	3.3092	2.1612	2.1593
1.25	0.1	34.578	34.575	1.2113	1.2117	1.2255	1.2258	1.1556	1.1561
	0.3	96.858	96.855	1.7476	1.7491	1.9611	1.9631	1.5231	1.5238
	0.5	159.57	159.59	2.5576	2.5592	3.9762	3.9803	2.1992	2.1705
1.5	0.1	34.578	34.576	1.2054	1.2057	1.2326	1.2329	1.1528	1.1531
	0.3	96.874	96.874	1.6808	1.6818	2.1114	2.1108	1.5090	1.5098
	0.5	160.14	160.64	2.3106	2.3105	7.8857	7.7554	2.1335	2.2038

In Tables 2-4 different composites with rhombic periodic cell ($w_1 = 1$, $w_2 = e^{i\theta}$) are considered. Table 2-3 show the in-plane and Table 4 the out-of-plane coefficients vs. fiber volume fraction. These composites belong to the monoclinic class of symmetry. For example, the effective properties C_{11}^* , C_{22}^* are different and the coefficients C_{16}^* , C_{26}^* , C_{36}^* are non-zero in comparison with hexagonal and tetragonal classes of composite. All coefficients are monotonic functions with respect to the fiber volume fraction for these composites. However, this behavior is not the same for variation of the angle with fixed fiber volume fraction. For instance, in Table 2, C_{11}^* is monotonic, whereas C_{66}^* is concave downward with respect to the angle. In Table 3, the composites exhibit a monoclinic global behavior, because C_{16}^* , C_{26}^* , C_{36}^* are non-zero, and they vanish for hexagonal and tetragonal symmetric classes. In order to validate FEM for the out-of-plane coefficients, Table 4 shows the dependency of the values of the effective coefficients $C_{55}^* / C_{44}^{(1)}$, $C_{45}^* / C_{44}^{(1)}$, $C_{44}^* / C_{44}^{(1)}$ for two given ratios of the shear modulus $\rho = C_{44}^{(2)} / C_{44}^{(1)}$, three different magnitudes of the volume fraction V_2 and a fixed angle of the periodic cell $\theta = \arccos(1/4)$. A comparison between the results reported by Golovchan and Nikityuk

(1981) (G&N), EEVM and AHM in Rodríguez-Ramos et al. (2011) and FEM in Würkner et al. (2011) is given. A good agreement between the aforementioned approaches can be noticed.

Table 2: Comparison of the effective properties between AHM and FEM with rhombic periodic cells

Angle θ	V_2	C_{11}^*		C_{12}^*		C_{22}^*		C_{33}^*	
		AHM	FEM	AHM	FEM	AHM	FEM	AHM	FEM
30°	0.1	4.0376	4.0384	1.6980	1.6977	4.1106	4.1123	34.579	34.576
	0.2	4.6551	4.6567	1.9075	1.9047	5.0999	5.1189	65.701	65.700
	0.3	5.3531	5.3559	2.1539	2.1554	7.0091	7.2488	96.920	96.940
50°	0.1	4.0494	4.0505	1.7145	1.7149	4.0572	4.0582	34.578	34.577
	0.3	5.5696	5.5757	2.3232	2.3204	5.7018	5.7117	96.853	96.851
	0.5	8.1740	8.1918	3.3977	3.3528	8.9916	9.1903	159.46	159.46
70°	0.1	4.0609	4.0615	1.7075	1.7074	4.0595	4.0600	34.578	34.577
	0.3	5.7209	5.7248	2.2425	2.2409	5.6993	5.7027	96.851	96.850
	0.5	8.9147	8.9571	3.0557	3.0137	8.7726	8.8087	159.44	159.44

Table 3: Comparison of the effective properties between AHM and FEM with rhombic periodic cells

Angle θ	V_2	C_{16}^*		C_{26}^*		C_{36}^*		C_{66}^*	
		AHM	FEM	AHM	FEM	AHM	FEM	AHM	FEM
30°	0.1	0.0124	0.0132	-0.0335	-0.0337	-0.0063	-0.0061	1.1616	1.1622
	0.2	0.0416	0.4346	-0.1699	-0.1783	-0.0385	-0.0404	1.3629	1.3660
	0.3	0.0635	0.0627	-0.5416	-0.6108	-0.1434	-0.1643	1.6642	1.6868
50°	0.1	0.0056	0.0054	-0.0103	-0.0098	-0.0014	-0.0013	1.1722	1.1725
	0.3	0.0479	0.0487	-0.1267	-0.1300	-0.0236	-0.0244	1.6870	1.6928
	0.5	0.0753	0.0755	-0.5624	-0.6693	-0.1461	-0.1778	2.7050	2.8006
70°	0.1	-0.0038	-0.0038	0.0057	0.0060	0.0006	0.0008	1.1651	1.1653
	0.3	-0.0378	-0.0379	0.0674	0.0681	0.0089	0.0090	1.6085	1.6104
	0.5	-0.0987	-0.1014	0.2939	0.3089	0.0586	0.0622	2.4261	2.4466

Table 4: Comparison of shear effective properties between different approaches with rhombic periodic cell and $\theta = \arccos(1/4)$

$C_{44}^{(2)} / C_{44}^{(1)}$	V_2	$C_{55}^* / C_{44}^{(1)}$				$C_{45}^* / C_{44}^{(1)}$				$C_{44}^* / C_{44}^{(1)}$			
		G&N	AHM	EEVM	FEM	G&N	AHM	EEVM	FEM	G&N	AHM	EEVM	FEM
20	0.3	1.75	1.75	1.75	1.75	0.02	0.02	0.02	0.02	1.74	1.74	1.74	1.74
	0.5	2.70	2.70	2.70	2.71	0.08	0.08	0.08	0.08	2.66	2.66	2.66	2.67
	0.7	5.00	5.00	5.00	4.99	0.34	0.34	0.34	0.35	4.83	4.83	4.83	4.82
120	0.3	1.84	1.84	1.84	1.85	0.02	0.02	0.02	0.02	1.83	1.83	1.83	1.84
	0.5	3.01	3.01	3.01	3.02	0.11	0.11	0.11	0.11	2.96	2.96	2.96	2.97
	0.7	6.47	6.47	6.47	6.43	0.60	0.60	0.60	0.60	6.16	6.16	6.16	6.15

In Tables 5-6 different composites with parallelogram periodic cell are studied. The parallelogram cell is characterized by $w_1=1$ and $\theta = \arccos(0.5/|w_2|)$ with varying of $|w_2|$. Table 5 shows selected in-plane properties calculated by FEM and AHM vs. fiber volume fraction, whereas Table 6 illustrates out-of-plane effective coefficients by Golovchan and Nikityuk (1981) (G&N) and the present models. A good agreement between the three aforementioned approaches can be stated.

Table 5: Comparison of the in-plane effective coefficients with parallelogram periodic cells for different ratios $|w_2|/|w_1|$ and fiber volume fractions

$ w_2 / w_1 $	V_2	C_{11}^*		C_{12}^*		C_{22}^*		C_{66}^*	
		AHM	FEM	AHM	FEM	AHM	FEM	AHM	FEM
1.1	0.1	4.0621	4.0651	1.7086	1.7091	4.0561	4.0585	1.1657	1.1671
	0.3	5.7327	5.7547	2.2659	2.2535	5.6394	5.6557	1.6077	1.6200
	0.5	8.9137	9.1901	3.2075	3.0613	8.3894	8.4835	2.3723	2.4609
1.25	0.1	4.0741	4.0755	1.7026	1.7010	4.0565	4.0594	1.1607	1.1616
	0.3	5.9190	5.9508	2.1924	2.1829	5.6245	5.6339	1.5625	1.5701
	0.5	10.0040	10.6754	2.9316	2.7725	8.2009	8.2491	2.256	2.3045
1.4	0.1	4.0861	4.0889	1.6978	1.6991	4.0551	4.0572	1.157	1.1578
	0.3	6.1396	6.1886	2.1311	2.1195	5.5872	5.5945	1.5344	1.5409
	0.5	11.7480	13.4746	2.6968	2.5605	7.9427	7.9665	2.2087	2.2340

Table 6: Comparison of the out-of plane effective coefficients with parallelogram periodic cells for different ratios $|w_2|/|w_1|$ and fiber volume fractions

$ w_2 / w_1 $	V_2	$C_{55}^* / C_{44}^{(1)}$			$C_{44}^* / C_{44}^{(1)}$		
		G&N	AHM	FEM	G&N	AHM	FEM
1.1	0.3	1.87	1.8657	1.8678	1.81	1.8110	1.8130
	0.5	3.11	3.1088	3.1110	2.81	2.8088	2.8095
	0.6	4.33	4.3300	4.3334	3.63	3.6259	3.6273
1.25	0.3	1.92	1.9245	1.9255	1.77	1.7674	1.7688
	0.5	3.55	2.5288	2.5294	2.62	2.6195	2.6207
	0.6	5.90	5.8965	5.9107	3.26	3.2633	3.2667
1.4	0.3	2.00	1.9998	2.0032	1.72	1.7252	1.7258
	0.5	4.40	4.4001	4.4188	2.46	2.4578	2.4617
	0.6	19.11	19.108	24.670	2.98	2.9774	2.9837

Both approaches (AHM and FEM) numerically satisfy the universal relations given in

(2.11)-(2.12) and they are equal to the ratio $\frac{\|C_{1111} + C_{1122}\|}{\|C_{1133}\|} = 3.33$. This fact shows that the

numerical implementation is correct.

5. Conclusions

A fiber-reinforced composite having a monoclinic symmetry has been studied. The fibers have a circular cross-section and are periodically distributed in the matrix. They are in perfect contact with the matrix.

The focus in this paper was set to present algorithms for predicting the full set of effective coefficients for composites based on unit cells models, which have non-rectangular shapes.

In literature for such types of cells nearly only results can be found for selected coefficients namely out-of-plane coefficients.

Two procedures are shown FEM and AHM. They are applied to find the overall properties of the composite. In AHM closed-form expressions for all effective coefficients of the composite were obtained by solving six local problems, whose solution is based on potential methods of a complex variable and properties of Weierstrass elliptic and related functions. In the FEM calculations a modified homogenization algorithm was developed to overcome obstacles with non-rectangular cells. The composites exhibit monoclinic behavior (related to principal axes) for rhombic periodic cell. Composites with periodic reinforced rectangular arrays have an orthorhombic symmetry. However, parallelogram patterns have not in general orthotropic behavior. Moreover, composites with square and hexagonal cells behave as transversely isotropic materials. Furthermore, universal relations for fibrous composites are achieved for any configuration of double periodic arrays and both methods satisfy the connections.

Results are compared between both methods and for selected coefficients with values from literature. Good agreements could be found in nearly all cases.

Acknowledgement

The funding of Conacyt project number 129658 is gratefully acknowledged. This work was partially supported by the postgraduate program of the German Federal State of Saxony-Anhalt and by the Graduate College 1554 Micro-Macro-Interactions in structured Media and Particle Systems of the German Research Foundation. This support is gratefully acknowledged. Thanks to the project of Basic Science of CITMA, Cuba.

Appendix

$$H_1(h_n) = A_2 \chi_1 B R^2 \delta_{1n} + \frac{1}{2} A_1 (\bar{S}_{n+1} R^{n+1} + \bar{A}_1 R^2 \delta_{1n}) C R^2$$

$$H_2(h_n) = \bar{A}_3 B R^2 \delta_{1n} + \frac{1}{2} \bar{A}_1 (\bar{S}_{n+1} R^{n+1} + \bar{A}_1 R^2 \delta_{1n}) C R^2,$$

$$H_3(h_n) = \left[E \frac{Y_{1p}}{Y_{2p}} - \bar{A}_1 R^2 P \right] \delta_{1n} - \bar{S}_{n+1} R^{n+1} P, \quad Y = \begin{pmatrix} -\|m\| & \|m\| & 0 \\ \|k\| & \|k\| & \|l\| \end{pmatrix},$$

$$W(w_{kn}) = A r_{kn} + \frac{C(\bar{S}_{n+1} R^{n+1} + \bar{A}_1 R^2 \delta_{1n})}{2} k S_{k+1} R^{k+1},$$

$$M(m_{kn}) = B G_{kn} + \frac{C(\bar{S}_{n+1} R^{n+1} + \bar{A}_1 R^2 \delta_{1n})}{2} k \bar{S}_{k+1} R^{k+1}.$$

$$S_{n+k} = \sum_{s,t \ (s^2+t^2 \neq 0)} \frac{1}{(P_{st})^{n+k}}, \quad T_{n+k} = \sum_{s,t \ (s^2+t^2 \neq 0)} \frac{\bar{P}_{st}}{(P_{st})^{n+k+1}} \quad s, t - \text{integer numbers},$$

$$A = B \frac{\chi^* \chi_1 - \chi_2}{\chi^* + \chi_2}, \quad B = \frac{1 - \chi^*}{\chi^* \chi_1 + 1}, \quad C = B \frac{1 + \chi_1 \chi^* - \chi^* - \chi_2}{\alpha_0}, \quad E = -\frac{1}{\chi^* \chi_1 + 1}, \quad P = B \frac{\chi_2 - 1}{2\alpha_0},$$

$$\chi^* = \frac{m_2}{m_1}, \quad \chi_\alpha = 3 - 4v_\alpha, \quad v_\alpha = \frac{C_{12}^{(\alpha)}}{C_{11}^{(\alpha)} + C_{12}^{(\alpha)}}, \quad A_1 = \frac{\bar{w}_1 \delta_2 - \bar{w}_2 \delta_1}{\bar{w}_1 w_2 - w_1 \bar{w}_2}, \quad A_2 = \frac{\delta_1 w_2 - \delta_2 w_1}{\bar{w}_1 w_2 - w_1 \bar{w}_2},$$

$$A_3 = \frac{\gamma_1 \bar{w}_2 - \gamma_2 \bar{w}_1}{\bar{w}_1 w_2 - w_1 \bar{w}_2}, \quad \delta_\gamma = \zeta(z + w_\gamma) - \zeta(z), \quad \gamma_i = Q(z + w_i) - Q(z) - \bar{w}_i \wp(z),$$

$$G_{kp} = (p+2) \bar{\eta}_{k(p+2)} + k \bar{\eta}_{(k+2)p} + k \bar{g}_{kp}, \quad \eta_{kp} = -k \frac{(k+p-1)!}{k! p!} S_{k+p} R^{k+p},$$

$$\alpha_0 = \chi^* \left[1 - \text{Re}(A_2) R^2 \right] + (\chi_2 - 1) \left[\frac{\text{Re}(A_2) R^2}{\chi_1 - 1} + \frac{1}{2} \right],$$

$$r_{kp} \equiv \sum_{t=3}^{\infty} \eta_{kt} \bar{\eta}_p, \quad g_{kp} = R^{k+p} C_{k+p}^p T_{k+p}, \quad \beta_0 = (\chi_2 + 1) \left(\frac{\text{Re}(A_2) R^2}{\chi_1 - 1} + \frac{1}{2} \right) - i \chi^* \text{Im}(A_2) R^2,$$

$$\Delta_p = \frac{A_1(F_+^- - \chi_1 F_-^+)}{\chi_1 - 1} R^2 a_{1(pp)} + \frac{\bar{A}_1(F_-^+ - \chi_1 F_+^-)}{\chi_1 - 1} R^2 \bar{a}_{1(pp)} + F_+^+ \sum_{k=1}^{\infty} \eta_{k1}^* a_{k(pp)} + F_-^- \sum_{k=1}^{\infty} \bar{\eta}_{k1}^* \bar{a}_{k(pp)} + B(\chi_2 + 1) - 2\beta_0 P,$$

where

$F_y^x = B(\chi_2 + 1 + \text{sgn}(x)\chi^* \chi_1 + \text{sgn}(y)\chi^*) + C\beta_0$, $\text{sgn}(x) = +, -$ and the over bar means complex conjugate number.

References

- Berger, H., Sreedhar, K., Gabbert, U., Rodríguez-Ramos, R., Guinovart, R., Otero, J.A., & Bravo-Castillero, J. (2005). An analytical and numerical approach for calculating effective material coefficients of piezoelectric fiber composites. *International Journal of Solids and Structures*, 42, 5692-5714.
- Golovchan, V.T., & Nikityuk, N.I. (1981). Solution of the problem of shear of a fibrous composite medium. Translated from *Prikladnaya Mekhanika*, 17, 29-35.
- Guinovart-Díaz, R., Bravo-Castillero, J., Rodríguez-Ramos, R., & Sabina, F.J. (2001). Closed-form expressions for the effective coefficients of fiber-reinforced composite with transversely isotropic constituents—I. Elastic and hexagonal symmetry. *Journal of Mechanics of Physics and Solids*, 49, 1445-62.
- Guinovart-Díaz, R., López-Realpozo, J.C., Rodríguez-Ramos, R., Bravo-Castillero, J., Ramírez, M., Camacho-Montes, H., & Sabina, F.J. (2011). Influence of parallelogram cells in the axial behaviour of fibrous composite. *International Journal of Engineering Science*, 49, 75-84.

- Hill, R. (1964). Theory of mechanical properties of fibre-strengthened materials: I. Elastic behavior. *Journal of Mechanics of Physics and Solids* 12, 199-212.
- Molkov, B.A., & Pobedria, B.E. (1985). Effective characteristic of fibrous unidirectional composite with periodic structure. *Mechanics of Solids*, 2, 119–29 [in Russian].
- Rodríguez-Ramos, R., Sabina, F.J., Guinovart-Díaz, R., & Bravo-Castillero, J. (2001). Closed-form expressions for the effective coefficients of a fiber-reinforced composite with transversely isotropic constituents—I. Elastic and square symmetry. *Mechanics of Materials*, 33, 223-35.
- Rodríguez-Ramos, R., Yan, P., Lopez-Realpozo, J.C., Guinovart-Díaz, R., Bravo-Castillero, J., Sabina, F.J., & Jiang, C.P. (2011). Two analytical models for the study of periodic fibrous elastic composite with different unit cells. *Composite Structures*, 93, 709-14.
- Royer, D., & Dieulesaint, E. (2000). *Elastic waves in Solids*. Berlin, Heidelberg: Springer Verlag.
- Suquet, P. M. (1987). Elements of homogenization for inelastic solid mechanics. In E. Sanchez-Palencia & A. Zaoui (Eds.), *Homogenization techniques for composite media* (pp. 193–278). Berlin, Heidelberg: Springer-Verlag.
- Würkner, M., Berger, H., & Gabbert, U. (2011). On numerical evaluation of effective material properties for composite structures with rhombic fiber arrangements. *International Journal of Engineering Science*, 49, 322-32.