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Dear Editors,

I would like to submit the contribution entitled "Effective elastic properties of a periodic fiber reinforced composite with parallelogram-like arrangement of fibers and imperfect contact between matrix and fibers" by Raúl Guinovart-Díaz, Reinaldo Rodríguez-Ramos, Julián Bravo-Castillero, Juan C. López-Realpozo, Federico J. Sabina and Igor Sevostianov, in the journal International Journal of Solids and Structures.

I would greatly appreciate your asistance in this sense.

Thank you in advanced.

Best regards,

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Effective elastic properties of a periodic fiber reinforced composite with parallelogram-like arrangement of fibers and imperfect contact between matrix and fibers.

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Abstract

The paper focuses on application of asymptotic homogenization method (AHM) to calculation of the effective elastic constants for fiber reinforced periodic composite with imperfect contact conditions between fibers and matrix. The arrangement of the fibers is assumed to be parallelogram like and imperfectness of the contact is modeled by linear springs. This work is an extension of previously reported results of Lopez-Realpozo et al. (2011), where perfect contact between the phases of the composite with parallelogram cells has been considered. The constituents of the composite are assumed to possess co-axial transversely isotropic properties. The obtained results are compared

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Introduction

Most of the presently developed micromechanical models of fiber-reinforced composites operate with perfect fiber-matrix bond and assume continuity of displacements and normal stress across the interface. However, experiments show that local or partial debonding at interfaces is a rule rather than the exception in composites (Rokhlin et al, 1994; Hui-Zu and Tsu-Wei 1995). Another imperfectness is related to formation of the third phase during manufacturing process due to chemical treatments of fiber surfaces and partial resin crystallization (Achenbach and Zhu, 1989). To describe these imperfectnesses, several approaches have been developed in which the bond between the inhomogeneities and the matrix is modeled by an interphase zone of certain thickness and elastic properties.

Mathematical analyses of inhomogeneous interfaces started, probably, with the work of Kanaun and Kudryavtseva (1983, 1986) on the effective elasticity of a medium with spherical and cylindrical inclusions, correspondingly, surrounded by radially inhomogeneous interphase zones. In these papers, the basic idea of replacing an

inhomogeneous inclusion by an equivalent homogeneous one has been formulated. Such a replacement was carried out by modeling the inhomogeneous interface by a number of thin concentric layers. The effective elastic properties of a composite containing a finite concentration of inclusions was found by replacing inhomogeneous inclusions by equivalent homogeneous ones (with properties found by the multilayer approximation) and applying the effective field method, whereby each inclusion is placed in a certain effective stress.

The ideas developed by Kanaun and Kudryavtseva (1983, 1986) have appeared in a number of later works (see, for example, Herve and Zaoui 1993). The basic idea of replacing inhomogeneous inclusions by equivalent homogeneous ones has been utilized in the majority of works on the topic. Theocaris (1985) and Theocaris et al. (1986) considered the mesophase layer as an independent phase of variable properties, matching those of the inclusion onone side and the matrix on the other. The examples of laws of variation of the Young's modulus E and Poisson's ratio ν across the interphase layer include linear, parabolic, hyperbolic and a logarithmic ones. Theocaris and Varias (1986) described a model predicting the influence of the mesophase on effective properties of fibrous composites. Their model is based on a corrected version of Kerner's model (see Christensen, 1979). Pagano and Tandon (1988,1990) developed two models to approximate the thermoelastic response of a composite body reinforced by coated fibers oriented in various directions. The fundamental representative volume element used in these papers is a three-phase concentric circular cylinder subjected to prescribed displacements and surface tractions. The analysis leads to estimation of the effect of interphase layers on the effective thermoelastic properties of fiber reinforced composites.

Sutcu (1992) presented a simple recursive algorithm which accounts for two concentric cylinders at a time, in order to calculate five effective elastic constants and two linear thermal expansion coefficients for a uniaxially aligned composite that contains an arbitrary number of coatings on its fibers. The effective elastic properties are calculated using expressions for two phase composites proposed by Hashin (1979) and Christensen (1979). Dasgupta and Bhandarkar (1992) discussed a method to obtain the transversely isotropic effective thermomechanical properties of unidirectional composites reinforced with coated cylindrical fibers. In this work the method developed by Benveniste et al. (1989) is applied to composites with multiply-coated cylindrical inhomogeneities. Lagache et al. (1994) determined numerically the effect of a mesophase using a finite element formulation in order to solve the local problems derived from the homogenization method. Chu and Rokhlin (1995) suggested a method for the inverse determination of effective elastic moduli of mesophases using a multi-phase generalized self-consistent model.

The idea of approximating radially variable properties by multiple layers (piecewise constant variation of properties) was explored by Garboczi and Bentz (1997) and Garboczi and Berryman (2000) in the context of applications to concrete composites. An alternative method was used by Jasiuk et al (1989, 1992) and Wang and Jasiuk (1998). They considered a general composite material with spherical inclusions representing the interphase as a functionally graded material and calculated effective elastic moduli using the composite spheres assemblage method for the effective bulk modulus and the generalized self-consistent method for the effective shear modulus.

Several closed form solutions for two specific forms of radial variation of properties – the linear and the power law ones – have been produced. Lutz and Ferrari (1993) and Zimmerman and Lutz (1999) considered inclusions with linearly varying elastic moduli,

in the context of the effective bulk modulus. Lutz and Zimmerman (1996a) considered the linear variation of the thermal expansion coefficient. Lutz and Zimmerman (1996b, 2005) and Lutz et al. (1997) considered the power law variation, in the context of effective bulk modulus and effective conductivity.

Alternative approach has been proposed by Hashin (1991a,b; 2002). He suggested to consider the interphase as a boundary layer of zero thickness with discontinuity of the displacements across this layer. More recently, Caporale et al. (2006) used finite elements analysis to study mechanical behavior of unidirectional fiber reinforced composites with imperfect interfacial bonding. In this work, an interfacial failure model is implemented by connecting the fibers and the matrix at the finite element nodes by normal and tangential brittle-elastic springs. Bisegna and Caselli (2008) and Artioli et al (2010) obtained closed-form expression for the homogenized longitudinal shear moduli of a linear elastic composite material reinforced by long, parallel, circular fibres with a periodic arrangement and imperfect (linear) fiber-matrix interface.

In the present paper we consider composite with parallelogram-like arrangement of fibers and discuss effect of the interface imperfectness on the overall elastic moduli. The analytical expressions are obtained by two scale asymptotic homogenization method (see books of Sanchez-Palencia, 1980; Pobedria,1984; Bakhalov and Panasenko,1989). This work continues previous study of Guinovart-Díaz et al. (2011) where only perfect contact has been considered and Sevostianov et al (2012), where square arrangement of fibers with special type of the interface imperfectness (incompressible layer between the phases) has been discussed. The novelty of the present analysis is that the formulation of the local problems for linear two phase elastic composites with spring imperfect contact conditions is given in general form. The solution for each plane local problems

is found using the potential methods of a complex variable and the properties of doubly periodic Weierstrass elliptic functions. The complet set of effective elastic constants is obtained in explicit formusing the asymptotic homogenization method (AHM) for fiber reinforced composites with periodic parallelogram cell of circular cylindrical shape periodically distributed in the matrix under imperfect contact conditions. The results are compoared with some numerical examples and with the results obtained by differential approach developed by Sevostianov (2007) and Sevostianov and Kachanov (2007) extended to parallelogram periodic cell. In this case, the isotropic interphase properties are related to the thickness, volume fraction and the spring imperfect parameters.

Statement of the problem. Local problems based on asymptotic homogenization method

We consider a two phase linear elastic materials with both phases being transversely-isotropic; the axis of transverse symmetry coincides with the fiber direction, which is taken as the Ox_3 axis. The fibers of circular cross-section are periodically distributed without overlapping in directions parallel to the Ow_1-and Ow_2-axis, where $w_1 \neq 0$ and $w_2 \neq 0 \left(w_2 \neq \lambda w_1,\ \lambda \in \mathbb{R}\right)$ are two complex numbers which define the parallelogram periodic cell of the two-phase composite. As shown in Fig. 1, the infinitely extended doubly-periodic structure is obtained from an elementary cell which is repeated in the two directions with fundamental periods w_1 and w_2. The general period P_{st} can be defined as $P_{st} = s\,w_1 + t\,w_2$, where s and t are arbitrary integers. The fibers and the matrix are not perfectly bonded. The mechanical behavior of imperfect interface is

modeled via a layer of mechanical springs of zero thickness. The spring constants $\tilde{K}_n$, $\tilde{K}_t$ and $\tilde{K}_s$ that have dimension of stress divided by length are the measures for the magnitude of the associated continuities on Γ; indices n, t and s denote normal and tangential directions on the interface Γ. Hereafter, these constants are called *interface parameters*. The perfect bonding interface is achieved when the values of these constants approach infinity. The vanishing value of $\tilde{K}_n$, $\tilde{K}_t$ and $\tilde{K}_t$ correspond to pure debonding (normal perfect debonding), in-plane pure sliding, and out-of-plane pure sliding, respectively. Any finite position values of the interface parameters define an imperfect interface. Within this approach the composite is modeled as a two phase material with imperfect interface conditions. The status of the mechanical bonding is completely determined by appropriate values of these constants.

Using the vector notation and defining the spring stiffness matrix, the displacement and the traction vectors as

$$\mathbf{u} = \begin{pmatrix} u_n \\ u_t \\ u_s \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} T_n \\ T_t \\ T_s \end{pmatrix}, \quad \tilde{\mathbf{K}} = \begin{pmatrix} K_n & 0 & 0 \\ 0 & K_t & 0 \\ 0 & 0 & K_s \end{pmatrix}, \tag{1}$$

the mechanical imperfect condition (Hashin 1990, Shodja et al. 2006) in general may be expressed in the following form

$$\mathbf{T}^{(1)} + \mathbf{T}^{(2)} = 0, \quad \mathbf{T}^{(\gamma)} = (-1)^{\gamma+1} \tilde{\mathbf{K}} \|\mathbf{u}\|, \text{ on } \Gamma. \tag{2}$$

In these relations $\|\bullet\|$ indicates the jump in the quantity across the interface Γ between the fiber and the matrix; u_n, u_t, u_s are normalandtangential components of the displacement vector; T_n, T_t, T_s are normal and tangential components of the traction vector $\mathbf{T}$ ($T_i = \sigma_{ij} n_j$) and $\mathbf{n}$ is the outward unit normal on Γ. The superscripts (γ), $\gamma = 1, 2$ denote the matrix and fiber respectively.

Fig. 1. The heterogeneous medium and the parallelogram periodic cell

Now, the overall properties of the above periodic medium can be calculated using the asymptotic homogenization method. In terms of the fast variable $\mathbf{y}$, the periodic cell S is taken as a parallelogram in the y_1y_2-plane so that $S = S_1 \cup S_2$ with $S = S_1 \cap S_2 = \emptyset$, where the domain S_2 is occupied by the matrix and its complement S_1 (fiber) is a circle of radius R centered at the origin O (Fig. 1). The common interface between the fiber and the matrix is denoted by Γ.

To obtain a system of equations for displacements, we susbstitute the constitutive equation

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \tag{3}$$

(where $\sigma_{ij}, \varepsilon_{ij}$ are the stress and strain tensors and C_{ijkl} are components of the elastic stiffness tensor) and strain-displacement relation into equilibrium equation. It yields a system of partial differential equations for u_i with rapidly oscillating coefficients

$$\left(C_{ijkl}(\mathbf{y})\,u^{\varepsilon}_{k,l}(\mathbf{x})\right)_{,j} = 0, \text{ in } \Omega. \tag{4}$$

The boundary and interface conditions are

$$u^{\varepsilon}_i = u^0_i \text{ on } \partial\Omega_u; \quad \sigma^{\varepsilon}_{ij} n_j = S^0_i \text{ on } \partial\Omega_T, \tag{5}$$

where $\partial\Omega \setminus \partial\Omega_u = \partial\Omega_T$, and u^0_i, S^0_i are the prescribed displacements and tractions on the external boundary. In order to study the imperfect contact conditions, the relations

between the displacement and traction vectors (1) are related to their Cartesian representations by the following expressions,

$$\begin{pmatrix} u_n \\ u_t \\ u_s \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \begin{pmatrix} T_n \\ T_t \\ T_s \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}. \tag{6}$$

Thus, the expression (2) on Γ, can be rewritten in the following indicial form,

$$\mathbf{T}^{(1)} + \mathbf{T}^{(2)} = 0, \tag{7}$$

$$T_n^{(\gamma)} = (-1)^{\gamma+1} K_n \|u_n\|,\ T_t^{(\gamma)} = (-1)^{\gamma+1} K_t \|u_t\|,\ T_s^{(\gamma)} = (-1)^{\gamma+1} K_s \|u_s\|. \tag{8}$$

Using AHM, it is possible to obtain an asymptotic solution of the boundary-value problem (4)-(8) as $\varepsilon \to 0$. The solution has the form of a series in powers of ε with coefficients depending on both the variables $\mathbf{x}$ and $\mathbf{y} = \mathbf{x}/\varepsilon$ treated as independent; they are referred as the slow or macroscopic and fast or microscopic variables, respectively, where ε = l/L is a small dimensionless parameter and L is a linear dimension of the body. Here, the solution is explicitly posed as

$$\mathbf{u}^{\varepsilon}(\mathbf{x}) = \mathbf{u}^{(0)}(\mathbf{x}) + \varepsilon \mathbf{u}^{(1)}(\mathbf{x},\mathbf{y}) + O(\varepsilon^2), \tag{9}$$

where $\mathbf{u}^{(0)}$ satisfy the homogenized system of differential equations

$$C^*_{ijlk} u^{(0)}_{l,kj} = 0. \quad \text{on } \Omega, \tag{10}$$

and the asterisk denotes the overall elastic properties. The term $\mathbf{u}^{(1)}$ represent a correction of the $\mathbf{u}^{(0)}$. Function $\mathbf{u}^{(1)}$ is found in combination of function $\mathbf{M}(\mathbf{y})$ representing solution of the local problemand the partial derivatives of $\mathbf{u}^{(0)}$.

Then, the original constitutive relations with rapidly oscillating material coefficients (4) are transformed into equivalent system (10) with constant coefficients $\mathbf{C}^*$ which represent the effectiveelastic stiffnesses of the equivalent homogeneous medium.

The main problem to obtain average coefficients is to find the periodic solutions of six ${}_{pq}L$ local problems on S in terms of the fast variable $\mathbf{y}$, where $p,q=1,2,3$. Each local problem uncouples into sets of equations, i.e. plane-strain and antiplane-strain systems. Table 1 shows the correspondence between the effective properties and the local problems. The global behavior of the composite is related to monoclinic class symmetry (Royer and Dieulesaint 2000) which contains 13 different elastic coefficients.

Table 1. Effective properties related to the local problems

In the present work, the ${}_{pq}L$ problem corresponds to finding displacements ${}_{pq}M^{(\gamma)}(\mathbf{y})$ in S_γ, $\gamma=1,2$ (double periodic functions with w_1, w_2) from the following system of partial differential equations

$$ {}_{pq}\sigma^{(\gamma)}_{i\delta,\delta}=0 \text{ in } S_\gamma, \tag{11}$$

where

$$ {}_{pq}\sigma^{(\gamma)}_{i\delta}=C^{(\gamma)}_{i\delta k\lambda}\ {}_{pq}M^{(\gamma)}_{k,\lambda}, \tag{12}$$

the comma notation denotes a partial derivative relative to the y_δ component (i.e., $U_{,\delta}\equiv\partial U/\partial y_\delta$).

Thus the expression (7)-(8) on Γ for the ${}_{pq}L$ problem can be expressed in the following indicial form,

$$ {}_{pq}\mathbf{T}^{(1)}+{}_{pq}\mathbf{T}^{(2)}=0, \tag{13}$$

$$ {}_{pq}T^{(\gamma)}_{n}=(-1)^{\gamma+1}K_n\left\|{}_{pq}M_n\right\|, \tag{14}$$

$$ {}_{pq}T^{(\gamma)}_{t}=(-1)^{\gamma+1}K_t\left\|{}_{pq}M_t\right\|, \tag{15}$$

$$ {}_{pq}T_s^{(\gamma)} = (-1)^{\gamma+1} K_s \left\| {}_{pq}M_s \right\|, \tag{16} $$

where M_n, M_t, M_s and T_n, T_t, T_s have the same meaning as (1) but adequate to ${}_{pq}L$ problems. To assure the solution of the ${}_{pq}L$ problems is unique, the functions also satisfy condition that $\langle {}_{pq}M_k \rangle = 0,$ where the angular brackets define the volume average per unit length over the unit periodic cell ($\langle F \rangle = \frac{1}{|S|}\int_S F(\mathbf{y})\,d\mathbf{y}.$). The symmetry between the indices p and q shows that at most six problems need to be considered. Once the local problems are solved, the homogenized moduli C^*_{ijpq} may be from the following formulae:

$$ C^*_{ijpq} = \langle C_{ijpq} + C_{ijkl\ pq}M_{k,l} \rangle . \tag{17} $$

The potential method of complex variables $z = y_1 + iy_2,\ (y_1, y_2) \in S$ and the properties of doubly periodic Weierstrass function $\wp(z) = \frac{1}{z^2} + \sum_{s,t}{}' \left\{ \frac{1}{(z-P_{st})^2} - \frac{1}{P_{st}^2} \right\}, P_{st} = s + it,\ s,t = 0, \pm 1, \pm 2, \ldots ,$ and related functions (Z- function $\zeta(z) = -\wp'(z)$ and Natanzon´s function $Q(z)$) are used for the solution of the local problems (11)-(16). Hence, the non-zero solution ${}_{pq}M_k^{(\alpha)}$ in S_α of the problem (11)-(16) must be found among doubly periodic functions of periods $w_1,\ w_2$ (see Fig. 1). Each local problem (11)-(16) uncouples into sets of equations. An in-plane strain system for ${}_{pq}M_\lambda^{(\alpha)},\ \lambda = 1, 2$ and an out-of-plane strain Laplace’s equation for ${}_{pq}M_3^{(\alpha)}$ has to be solved. Then the solutionsfor the in-plane (out-of-plane) strain problems involve the determination of the in-plane (out-of-plane) displacements, strains and stresses over each phase S_α of the composite. Due to the non-vanishing components of the elastic tensors $C_{ijpq}^{(\alpha)},$ the only non-homogeneous problems, that have

a non-zero solution, correspond to the four in-plane strain problems ${}_{jj}L$ and ${}_{12}L$, and the two out-of-plane strain ones ${}_{23}L$ and ${}_{13}L$. In that way the solutions for both (in-plane and out-of-plane) local problems lead to obtain the average coefficients of the composite given in Fig. 1 (6 ${}_{pq}L$ problems need to be considered due to the symmetry between p and q).

Solution of plane problem ${}_{jj}L$ and the effective coefficients

Let, in this section $U^{(\gamma)} = {}_{jj}M^{(\gamma)}$ and the jj presubindices are dropped from all relevant quantities. From the equations (11)-(16), using relation (6) we have

$$\sigma^{(\gamma)}_{\alpha\delta,\delta} = 0 \quad \text{in} \quad S_\gamma\,; \text{ where } \sigma^{(\gamma)}_{\alpha\delta} = C^{(\gamma)}_{\alpha\delta\beta\lambda}\, U^{(\gamma)}_{\beta,\lambda}, \tag{18}$$

$$\left\| \sigma^{(\gamma)}_{\alpha\delta} n_\delta \right\|_1 = -\left\| C_{\alpha\delta jj} \right\|_1 n_\delta \quad \text{on} \quad \Gamma, \tag{19}$$

$$\sigma^{(2)}_{11} n_1^2 + \sigma^{(2)}_{22} n_2^2 + C^{(2)}_{11jj} n_1^2 + C^{(2)}_{22jj} n_2^2 + 2\sigma^{(2)}_{12} n_1 n_2 = (-1)^{1+\gamma} \frac{K_n m_1}{R} \left[\left\| U^{(\gamma)}_2 \right\| n_1 + \left\| U^{(\gamma)}_1 \right\| n_2 \right] \text{ on } \Gamma, \tag{20}$$

$$\left[\sigma^{(\gamma)}_{11} - \sigma^{(\gamma)}_{22} + \left(C^{(\gamma)}_{11jj} - C^{(\gamma)}_{22jj} \right) \right] n_1 n_2 + \sigma^{(\gamma)}_{12} \left(n_2^2 - n_1^2 \right) = (-1)^{1+\gamma} \frac{K_t m_1}{R} \left[\left\| U^{(\gamma)}_2 \right\| n_1 - \left\| U^{(\gamma)}_1 \right\| n_2 \right] \text{ on } \Gamma, \tag{21}$$

where

$$\begin{aligned} \sigma^{(\gamma)}_{11} &= (k_\gamma + m_\gamma) U^{(\gamma)}_{1,1} + (k_\gamma - m_\gamma) U^{(\gamma)}_{2,2}, \\ \sigma^{(\gamma)}_{22} &= (k_\gamma - m_\gamma) U^{(\gamma)}_{1,1} + (k_\gamma + m_\gamma) U^{(\gamma)}_{2,2}, \\ \sigma^{(\gamma)}_{12} &= m_\gamma (U^{(\gamma)}_{1,2} + U^{(\gamma)}_{2,1}), \end{aligned} \tag{22}$$

and $\mathbf{K} = \begin{pmatrix} K_n & 0 & 0 \\ 0 & K_t & 0 \\ 0 & 0 & K_s \end{pmatrix} = \dfrac{R\tilde{\mathbf{K}}}{m_1}$ is a matrix dimensionless parameters. Parameter $k = \left(C_{1111} + C_{1122} \right)/2$ is the plane-strain bulk modulus for lateral dilatation without longitudinal extension; $m = C_{1212} = \left(C_{1111} - C_{1122} \right)/2$ is the stiffness for longitudinal uniaxial strain.

It can be recognized that the structure of Eqs. (18) and (22) is one of plane-strain isotropic elasticity except that the usual Lame constants λ and μ are here identified with $k_\gamma - m_\gamma$ and m_γ, respectively. The method of complex variables in terms of two harmonic functions $\varphi_\gamma(z)$ and $\psi_\gamma(z)$ and the Kolosov-Muskhelishvili complex potentials can be applied. The potentials are related to the displacement and stress components by means of the classical formulae

$$2m_\gamma(U_1^{(\gamma)} + iU_2^{(\gamma)}) = \kappa_\gamma\varphi_\gamma(z) - z\bar{\varphi}_\gamma(z) - \bar{\psi}_\gamma(z), \tag{23}$$

$$\sigma_{11}^{(\gamma)} + \sigma_{22}^{(\gamma)} = 2(\varphi_\gamma^{'}(z) + \bar{\varphi}_\gamma^{'}(z)), \tag{24}$$

$$\sigma_{22}^{(\gamma)} - \sigma_{11}^{(\gamma)} + 2i\sigma_{12}^{(\gamma)} = 2(\bar{z}\varphi_\gamma^{''}(z) + \psi_\gamma^{'}(z)), \tag{25}$$

where the prime denotes a derivative with respect to z, the overbar means complex conjugate and $\kappa_\gamma = 3 - 4\nu_\gamma^T$, where ν_γ^T is the transverse Poisson's ratio.

Using formulae (24)-(25), the contact conditions (19)-(21) are transformed according to the Kolosov-Muskhelishvili complex potentials in the following form

$$z\overline{\varphi_1^{/}}(z) + \overline{\psi_1}(z) + \varphi_1(z) + \bar{z}\gamma_{1j} + z\gamma_{2j} = z\overline{\varphi_2^{/}}(z) + \overline{\psi_2}(z) + \varphi_2(z), \tag{26}$$

$$\begin{aligned}
&2R\chi_m e^{2i\theta}\left[\bar{z}\varphi_\alpha^{//}(z) + \psi_\alpha^{/}(z)\right] + 2R\chi_m e^{-2i\theta}\left[z\overline{\varphi_\alpha^{//}}(z) + \overline{\psi_\alpha^{/}}(z)\right] - \\
&\quad - 4R\chi_m\left[\varphi_\alpha^{/}(z) + \overline{\varphi_\alpha^{/}}(z)\right] - 4R\chi_m\gamma_{4j} + 2R\chi_m\left(e^{2i\theta} + e^{-2i\theta}\right)\gamma_{3j} = \\
&= -K_n\left\{\chi_m\left[\chi^{(1)}\varphi_1(z) - z\,\overline{\varphi_1^{/}}(z) - \overline{\psi_1}(z)\right] - \left[\chi^{(2)}\varphi_2(z) - z\,\overline{\varphi_2^{/}}(z) - \overline{\psi_2}(z)\right]\right\}e^{-i\theta} + \\
&\quad - K_n\left\{\chi_m\left[\chi^{(1)}\overline{\varphi}_1(z) - \bar{z}\varphi_1^{/}(z) - \psi_1(z)\right] - \left[\chi^{(2)}\overline{\varphi}_2(z) - \bar{z}\varphi_2^{/}(z) - \psi_2(z)\right]\right\}e^{i\theta},
\end{aligned} \tag{27}$$

$$\begin{aligned}
&2R\chi_m\left[z\overline{\varphi_2^{//}}(z) + \overline{\psi_2^{/}}(z)\right]e^{-i\theta} - 2R\chi_m\left[\bar{z}\varphi_2^{//}(z) + \psi_2^{/}(z)\right]e^{3i\theta} - 2\gamma_{3j}R\chi_m\left(e^{3i\theta} - e^{-i\theta}\right) = \\
&\quad = -K_t\left\{\begin{matrix}\chi_m\left[\kappa_1\varphi_1(z) - z\overline{\varphi_1^{/}}(z) - \overline{\psi_1}(z)\right] - \left[\kappa_2\varphi_2(z) - z\overline{\varphi_2^{/}}(z) - \overline{\psi_2}(z)\right] - \\ \chi_m\left[\kappa_1\overline{\varphi_1}(z) - \bar{z}\varphi_1^{/}(z) - \psi_1(z)\right]e^{2i\theta} - \left[\kappa_2\overline{\varphi_2}(z) - \bar{z}\varphi_2^{/}(z) - \psi_2(z)\right]e^{2i\theta}\end{matrix}\right\},
\end{aligned} \tag{28}$$

where

$$\chi_m = m_2 / m_1,\ 2\gamma_{1j} = C^{(1)}_{22jj} - C^{(2)}_{22jj} + C^{(2)}_{11jj} - C^{(1)}_{11jj},\ 2\gamma_{2j} = C^{(1)}_{11jj} - C^{(2)}_{11jj} + C^{(1)}_{22jj} - C^{(2)}_{22jj},$$

$$2\gamma_{3j} = C^{(2)}_{22jj} - C^{(2)}_{11jj},\ 2\gamma_{4j} = C^{(2)}_{22jj} + C^{(2)}_{11jj}. \tag{29}$$

The complex potentials $\varphi_\gamma(z)$ and $\psi_\gamma(z)$ are considered for the periodic cell that contains the origin of coordinates in the following form

$$\begin{aligned}\varphi_1(z) &= a_0 z + \sum_{k=1}^{\infty}{}^{o} \frac{a_k \zeta^{(k-1)}(z)}{(k-1)!},\\ \psi_1(z) &= b_0 z + \sum_{k=1}^{\infty}{}^{o} \frac{b_k \zeta^{(k-1)}(z)}{(k-1)!} + \sum_{k=1}^{\infty}{}^{o} \frac{a_k Q^{(k-1)}(z)}{(k-1)!},\end{aligned} \tag{30}$$

(for the matrix phase) and

$$\varphi_2(z) = \sum_{k=1}^{\infty}{}^{o}\ c_k z^k, \quad \psi_2(z) = \sum_{k=1}^{\infty}{}^{o}\ d_k z^k, \tag{31}$$

(for the fiber phase).

In the foregoing definitions $\zeta^{(k-1)}(z)$ denotes the doubly–periodic $(k-1)$-th derivative of the Weierstrass Zeta quasi–periodic function $\zeta(z)$, $\wp(z) = -\zeta'(z)$, $Q^{(k-1)}(z)$ is the doubly–periodic $(k-1)$-th derivative of Natanzon's quasi–periodic function, and the symbol $\sum^{o}$ indicates a sum over the indices k =1, 3, 5,..., meanwhile constants a_0, b_0, a_k, b_k, c_k and d_k (k = 1, 3, 5, . . .) are all complex and undefined.

Next, accounting for the double periodicity of matrix displacements $U^{(1)}$, using Legendre's relation and the properties of the doubly periodic functions, coefficients a_0 and b_0 can be written as follows

$$a_0 = \bar{A}_1 R^2 \bar{a}_1 + \frac{A_2 + \bar{A}_2 \kappa_1}{(\kappa_1 - 1)(\kappa_1 + 1)} R^2 b_1, \quad b_0 = A_3 R^2 a_1 + \bar{A}_2 \kappa_1 R^2 \bar{a}_1 - A_1 R^2 b_1, \tag{32}$$

where $A_1 = \dfrac{\bar{w}_1\delta_2 - \bar{w}_2\delta_1}{\bar{w}_1 w_2 - w_1 \bar{w}_2}$, $A_2 = \dfrac{\delta_1 w_2 - \delta_2 w_1}{\bar{w}_1 w_2 - w_1 \bar{w}_2}$ and $A_3 = \dfrac{\gamma_1\bar{w}_2 - \gamma_2\bar{w}_1}{\bar{w}_1 w_2 - w_1\bar{w}_2}$,

$$\delta_i = \zeta(z + w_i) - \zeta(z),\ \gamma_i = Q(z + w_i) - Q(z) - \bar{w}_i\wp(z), \quad i = 1, 2.$$

Replacing (30)-(31) into (26)-(28) the following relations between the unknown constants of the above expansions are obtained

$$b_1 = \frac{2C}{B}\Re e\left\{-R^2 A_1 a_1 + \sum_{k=1}^{\infty}{}^{o}\eta_{k1} a_k\right\} - \frac{P}{B} R\gamma_{2\beta}, \tag{33}$$

$$b_{p+2} = \left[p - \frac{D_p}{E_p} K_n K_t \chi_m (\kappa_1 + 1)\right] a_p - \left[1 + \frac{B_p}{E_p} K_n K_t \chi_m (\kappa_1 + 1)\right] \sum_{k=1}^{\infty}{}^{o}\overline{\eta}_{k\,p+2} \overline{a}_k, \tag{34}$$

$$c_1 = \frac{1}{2(\kappa_2 + 1)}\left(\begin{array}{r} -C_1^+ A_1 R^2 a_1 - C_1^- \overline{A}_1 R^2 \overline{a}_1 + C_1^+ \sum_{k=1}^{\infty}{}^{o}\eta_{k1} a_1 + \\ + C_1^- \sum_{k=1}^{\infty}{}^{o}\overline{\eta}_{k1} \overline{a}_1 + \left(\chi^{(2)} + 1 - 2\beta_0 \frac{P}{B}\right) R\gamma_{2j} \end{array}\right), \tag{35}$$

$$c_{p+2} = -\frac{K_n K_t \chi_m (\kappa_1 + 1)}{E_p}\left(D_p \overline{a}_p + B_p \sum_{k=1}^{\infty}{}^{o}\eta_{k\,p+2} a_k\right), \tag{36}$$

$$\begin{aligned} d_p = {} & K_n K_t \chi_m (\kappa_1 + 1)\left(\frac{C_p}{E_p}\overline{a}_p + \frac{A_p}{E_p}\sum_{k=1}^{\infty}{}^{o}\eta_{k\,p+2} a_k\right) + \\ & + \frac{C_p (K_n - K_t)\gamma_{3j} - A_p (K_n + K_t)\gamma_{3j} + K_n K_t C_p \gamma_{1j}}{E_p}\chi_m R\delta_{1p}, \end{aligned} \tag{37}$$

where

$$B = \left(1 - \chi_m - \frac{2p\chi_m}{K_n}\right)\left(A_p^0\right)^{-1}, \quad C = B\frac{1 - \kappa_2 + \chi_m (\kappa_1 - 1)}{2\alpha_0} + \frac{2\chi_m}{\alpha_0 K_n},$$

$$P = B\left(\frac{\kappa_2 - 1}{2\alpha_0} - \frac{4\chi_m \left(1 + \gamma_{4\beta}/\gamma_{2\beta}\right)}{2\alpha_0 K_n}\right), \quad C_1^{\pm} = 1 + \kappa_2 \pm \chi_m (1 + \kappa_1) + 2\beta_0 \frac{C}{B},$$

$$\alpha_0 = \chi_m\left[1 - \Re e\{A_2\} R^2\right] + \left(\kappa_2 - 1 - \frac{4\chi_m}{K_n}\right)\left[\frac{\Re e\{A_2\} R^2}{\kappa_1 - 1} + \frac{1}{2}\right],$$

$\beta_0 = \dfrac{(\kappa_2 + 1)\Re e\{A_2\} R^2}{\kappa_1 - 1} - i\chi_m \Im m\{A_2\} R^2 + \dfrac{(\kappa_2 + 1)}{2}$, $\Re e\{z\}$ and $\Im m\{z\}$ denote real and imaginary part of complex number z. Then,

$$A_p^0 = \chi_m \kappa_1 + 1 - \frac{2p\chi_m}{K_n} + K_n K_t \chi_m (\kappa_1 + 1)\left(\kappa_2 + \chi_m - \frac{2(p+2)\chi_m}{K_n}\right)\frac{D_p}{E_p},$$

$$A_p=(p+2)\left(K_nK_t(1-\chi_m)+p(K_n-K_t)\chi_m+(K_n+K_t)\chi_m\right),$$

$$B_p=K_tK_n(1-\chi_m)+p(K_n-K_t)\chi_m,$$

$$C_p=K_nK_t(\kappa_2+\chi_m)+(p+2)\chi_m\left((K_n+K_t)p+K_n-K_t\right),$$

$$D_p=p\chi_m(K_n+K_t),\ \ E_p=A_pD_p-C_pB_p,$$

δ_{ik} is the Kronecker's delta function, C_k^n is the binomial coefficient, V_1, V_2 are the area of the matrix and fiber respectively, $V_1+V_2=1$. The parameters involved in this expression are as follows

$$\eta_{kp}=-k\frac{(k+p-1)!}{k!\,p!}S_{k+p}R^{k+p},\ S_{n+k}=\sum_{p,q}\frac{1}{(p+iq)^{n+k}},\ \ T_{n+k}=\sum_{p,q}\frac{p-iq}{(p+iq)^{n+k+1}},$$

$p^2+q^2\neq 0,\ \ p,q-$integer numbers.

The above expression (33)-(37) depend on the unknown parameter a_p which can be calculated from the following system of algebraic equations

$$a_p+H_{1p}a_1+H_{2p}\bar{a}_1+\sum_{k=1}^{\infty}{}^{o}W_{kp}a_k+\sum_{k=1}^{\infty}{}^{o}M_{kp}\bar{a}_k=H_{3p}R\gamma_{2j},\tag{38}$$

where $p=1,3,5,\dots,$

$$H_{1p}=BA_2\chi^{(1)}R^2\delta_{1p}-\left(\bar{\eta}_{1p}-\bar{A}_1R^2\delta_{1p}\right)R^2A_1C,\ \ H_{2p}=B\bar{A}_3R^2\delta_{1p}-\left(\bar{\eta}_{1p}-\bar{A}_1R^2\delta_{1p}\right)R^2\bar{A}_1C,$$

$$M_{kp}=BG_{kp}-BD_p^0\bar{\eta}_{k+2\,p}+\left(\bar{\eta}_{1p}-\bar{A}_1R^2\delta_{1p}\right)C\bar{\eta}_{k1}+B_p^0\left(A_p^0\right)^{-1}\bar{\eta}_{k\,p+2},$$

$$W_{kp}=-BC_k^0r_{kp}+\left(\bar{\eta}_{1p}-\bar{A}_1R^2\delta_{1p}\right)C\eta_{k1},$$

$$H_{3p}=\left(\bar{\eta}_{1p}-\bar{A}_1R^2\delta_{1p}\right)P+\frac{2\chi_m}{K_n}\left(A_p^0\right)^{-1}\frac{\gamma_{3j}}{\gamma_{2j}}\delta_{1p}-\left(1-\frac{2\chi_m p}{K_n}\right)\left(A_p^0\right)^{-1}\frac{\gamma_{1j}}{\gamma_{2j}}\delta_{1p},$$

$$B_p^0 = \chi_m\left(\kappa_1+1\right)\left[1+K_n K_t\left(\kappa_2+\chi_m-\frac{2\chi_m}{K_n}\left(p+2\right)\right)\frac{B_p}{E_p}\right],$$

$$C_p^0 = 1+K_n K_t \chi_m\left(1+\kappa_1\right)\frac{B_p}{E_p}, \quad D_p^0 = K_n K_t \chi_m\left(1+\kappa_1\right)\frac{D_p}{E_p},$$

$$r_{kp} = \sum_{i=3}^{\infty}{}^{o}\eta_{ki}\eta_{ip}, \; G_{kp} = \left(p+2\right)\eta_{k(p+2)} + k\eta_{k+2p} + kR^{p+k}C_{p+k}^{p}T_{p+k}.$$

Equation (38) represents an infinite linear system from which we can calculate coefficients a_p as, for example in in Rodríguez-Ramos et al. (2001).

In order to obtain analytic expressions of the effective elastic stiffnesses $C^*_{1111}, C^*_{1122}, C^*_{1133}$, $C^*_{2222}, C^*_{2233}, C^*_{3333}, C^*_{1211}, C^*_{1222}$, and C^*_{1233}, is necessary to apply the Green's theorem to (17) on the interface Γ, using the condition (23) and the orthogonality of the trigonometric functions. Then

$$C^*_{22\,jj} - C^*_{11\,jj} + 2iC^*_{12\,jj} = \left\langle C_{22\,jj} - C_{11\,jj}\right\rangle + \frac{2\pi R^2\gamma_{2j}}{V}\left[\left(\kappa_1+1\right)\frac{\bar{a}_{1j}}{R\gamma_{2j}} + \frac{\gamma_{1j}}{\gamma_{2j}}\right], \tag{39}$$

$$C^*_{11\,jj} + C^*_{22\,jj} = \left\langle C_{11\,jj} + C_{22\,jj}\right\rangle - 2V_2\frac{\|k\|\gamma_{2j}}{m_1}\Re e\left\{\kappa_2\Delta_j - \bar{\Delta}_j\right\} + 4V_2\frac{k_1\gamma_{2j}}{m_1 K_n}\left(2\chi_m \Re e\left\{\Delta_j\right\} + \frac{\gamma_{4j}}{\gamma_{2j}}\right), \tag{40}$$

$$C^*_{33\,jj} = \left\langle C_{33\,jj}\right\rangle - V_2\frac{\|C_{1133}\|\gamma_{2j}}{m_1}\Re e\left\{\kappa_2\Delta_j - \bar{\Delta}_j\right\} + 2V_2\frac{C^{(1)}_{1133}\gamma_{2j}}{m_1 K_n}\left(2\chi_m \Re e\left\{\Delta_j\right\} + \frac{\gamma_{4j}}{\gamma_{2i}}\right), \tag{41}$$

where $\Delta_j = c_1/\left(R\chi_m\gamma_{2j}\right)$, a_{1j} denotes the residue a_1 of the function $\varphi_1(z)$ in (30) for each problem ${}_{jj}L$.

The following analytical expressions of the effective properties are obtained replacing the previous equalities (29) into (39)-(41),

$$C^*_{1111}=\langle C_{1111}\rangle - V_2\frac{\|k\|^2}{m_1}\Re e\{\kappa_2\Delta_1-\bar{\Delta}_1\}-V_2\|k\|(\kappa_1+1)\Re e\left\{\frac{a_{11}}{R\|k\|}\right\}+V_2\|m\|+ \\ +2V_2\frac{k_1\|k\|}{m_1K_n}\left(2\chi_m\Re e\{\Delta_1\}+\frac{k_2}{\|k\|}\right), \tag{42}$$

$$C^*_{1122}=\langle C_{1122}\rangle - V_2\frac{\|k\|^2}{m_1}\Re e\{\kappa_2\Delta_2-\bar{\Delta}_2\}+V_2\|k\|(\kappa_1+1)\Re e\left\{\frac{a_{12}}{R\|k\|}\right\}-V_2\|m\|+ \\ +2V_2\frac{k_1\|k\|}{m_1K_n}\left(2\chi_m\Re e\{\Delta_2\}+\frac{k_2}{\|k\|}\right), \tag{43}$$

$$C^*_{1133}=\langle C_{1133}\rangle - V_2\frac{\|k\|\|C_{1133}\|}{m_1}\Re e\{\kappa_2\Delta_3-\bar{\Delta}_3\}-V_2\|C_{1133}\|(\kappa_1+1)\Re e\left\{\frac{a_{13}}{R\|C_{1133}\|}\right\}+ \\ +2V_2\frac{k_1\|C_{1133}\|}{m_1K_n}\left(2\chi_m\Re e\{\Delta_3\}+\frac{C^{(2)}_{1133}}{\|C_{1133}\|}\right), \tag{44}$$

$$C^*_{2222}=\langle C_{2222}\rangle - V_2\frac{\|k\|^2}{m_1}\Re e\{\kappa_2\Delta_2-\bar{\Delta}_{12}\}+V_2\|k\|(\kappa_1+1)\Re e\left\{\frac{a_{12}}{R\|k\|}\right\}+V_2\|m\|+ \\ +2V_2\frac{k_1\|k\|}{m_1K_n}\left(2\chi_m\Re e\{\Delta_2\}+\frac{k_2}{\|k\|}\right), \tag{45}$$

$$C^*_{2233}=\langle C_{2233}\rangle - V_2\frac{\|k\|\|C_{1133}\|}{m_1}\Re e\{\kappa_2\Delta_3-\bar{\Delta}_3\}+V_2\|C_{1133}\|(\kappa_1+1)\Re e\left\{\frac{a_{13}}{R\|C_{1133}\|}\right\}+ \\ +2V_2\frac{k_1\|C_{1133}\|}{m_1K_n}\left(2\chi_m\Re e\{\Delta_3\}+\frac{C^{(2)}_{1133}}{\|C_{1133}\|}\right), \tag{46}$$

$$C^*_{3333}=\langle C_{3333}\rangle - V_2\frac{\|C_{1133}\|^2}{m_1}\Re e\{\kappa_2\Delta_3-\bar{\Delta}_3\}+2V_2\frac{C^{(1)}_{1133}\|C_{1133}\|}{m_1K_n}\left(2\chi_m\Re e\{\Delta_3\}+\frac{C^{(2)}_{1133}}{\|C_{1133}\|}\right), \tag{47}$$

$$C^*_{1211}=V_2\|k\|\Im m\left[(\kappa_1+1)\frac{a_{11}}{R\|k\|}\right], \tag{48}$$

$$C^*_{1222}=V_2\|k\|\Im m\left[(\kappa_1+1)\frac{a_{12}}{R\|k\|}\right], \tag{49}$$

$$C^*_{1233}=V_2\|C_{1133}\|\Im m\left[(\kappa_1+1)\frac{a_{13}}{R\|C_{1133}\|}\right]. \tag{50}$$

The plane problem $_{12}\mathrm{L}$ and effective transverse shear

Let $U^{(\gamma)} = {}_{12}M^{(\gamma)}$, then, using (6) and equations (11)-(16), the local problem can be stated as follows

$$\sigma^{(\gamma)}_{\alpha\delta,\delta} = 0 \quad \text{in} \quad S_\gamma\,; \text{ where } \sigma^{(\gamma)}_{\alpha\delta} = C^{(\gamma)}_{\alpha\delta\beta\lambda}\, U^{(\gamma)}_{\beta,\lambda}, \tag{51}$$

$$\left\| \sigma^{(\gamma)}_{\alpha\delta} n_\delta \right\| = -\left\| C_{\alpha\delta 12} \right\| n_\delta \quad \text{on} \quad \Gamma, \tag{52}$$

$$\left[\sigma^{(\gamma)}_{11} - \sigma^{(\gamma)}_{22}\right] n_1 n_2 + \left(\sigma^{(\gamma)}_{12} + C^{(\gamma)}_{1212}\right)\left(n_2^2 - n_1^2\right) = (-1)^{1+\gamma} K_t \left[\left\| U^{(\gamma)}_2 \right\| n_1 - \left\| U^{(\gamma)}_1 \right\| n_2\right] \quad \text{on} \quad \Gamma, \tag{53}$$

$$\sigma^{(\gamma)}_{11} n_1^2 + \sigma^{(\gamma)}_{22} n_2^2 + 2\left(\sigma^{(\gamma)}_{12} + C^{(\gamma)}_{1212}\right) n_1 n_2 = (-1)^{1+\gamma}\, \tilde{K}_n \left[\left\| U_1 \right\| n_1 + \left\| U_2 \right\| n_2\right] \quad \text{on} \quad \Gamma. \tag{54}$$

Once the plane problems $_{jj}\mathrm{L}$ have been solved, it is easier to solve the plane problem $_{12}\mathrm{L}$. This can be realized if (52)-(54) are expressed in terms of the Kolosov-Muskhelishvili complex potentials (24)-(25). Consequently, we obtain similar expressions to (26)-(28) with $\gamma_{1j} = -i\left(C^{(1)}_{1111} - C^{(1)}_{2211} - C^{(2)}_{1111} + C^{(2)}_{2211}\right) = -i\left\| m \right\|$, $\gamma_{2j} = \gamma_{4j} = 0$, $\gamma_{3j} = -i(C^{(2)}_{1111} - C^{(2)}_{1112}) = -im_2$. The complex potentials $\varphi_\gamma(z)$ and $\psi_\gamma(z)$ have the same form as (30)-(31). The unknown constant a_p that appear now in the relation (30)-(31) can be obtained as the solution for the system of equations (38) where the right-hand side term $H_{3p}R\gamma_{2j}$ is now replaced by $-i\left\| m \right\| R\left[1 + \frac{2\chi_m}{K_n}\left(\frac{m_2}{\left\| m \right\|} - 1\right)\right]\left(A^0_p\right)^{-1}\delta_{1p}$. The remaining unknown constants a_0, b_0, b_p, c_p and d_p (p = 1, 3, 5, . . .) are obtained using relations (32)-(37).

The effective coefficients $C^*_{1112}, C^*_{2212}, C^*_{3312}, C^*_{1212}$ can be calculated from (17), applying the Green's theorem on the interface Γ, :

$$C^*_{1212} = C^{(1)}_{1212} - V_2 \left\| m \right\| (\kappa_1 + 1)\, \Im m\{a_1\}, \tag{55}$$

$$C^*_{1112} + C^*_{2212} = -2V_2 \frac{\|m\|\|k\|}{m_1} \Re e\{\kappa_2\Delta - \bar{\Delta}\} - 8V_2 \frac{\chi_m k_2 \|m\|}{K_n m_1} \Re e\{\Delta\}, \tag{56}$$

$$C^*_{2212} - C^*_{1112} = -2V_2 \|m\|(\kappa_1 + 1)\Re e\{a_1\}, \tag{57}$$

$$C^*_{3312} = -V_2 \frac{\|m\|\|C_{1133}\|}{m_1} \Re e\{\kappa_2\Delta - \bar{\Delta}\} - 4V_2 \frac{\chi_m C^{(2)}_{1133} \|m\|}{K_n m_1} \Re e\{\Delta\}, \tag{58}$$

where $\Delta = c_1 / (R\chi_m \|m\|)$ and a_1 denote the residue of the function $\varphi_1(z)$.

Finally, the antiplane problems ${}_{13}L$, ${}_{23}L$ and the effective axial shear moduli for parallelogram periodic cell have been previously derived by López-Realpozo et al. (2011) as follows:

$$C^*_{1313} = C^{(1)}_{1313}\left(1 - 2V_2\beta z_{22} / |Z|\right),$$

$$C^*_{2313} = C^*_{1323} = 2C^{(1)}_{1313} V_2 \beta z_{21} / |Z|, \tag{59}$$

$$C^*_{2323} = C^{(1)}_{1313}\left(1 - 2V_2\beta z_{11} / |Z|\right),$$

where $\beta = \dfrac{\left(C^{(1)}_{1313} - C^{(2)}_{1313}\right)K_s + C^{(2)}_{1313}}{\left(C^{(1)}_{1313} + C^{(2)}_{1313}\right)K_s + C^{(2)}_{1313}}$.

Formulae (42)-(50), (55) and (59) give explicit exact analytical solutions for the effective elastic stiffnesses $C^*_{1111}, C^*_{1122}, C^*_{1133}, C^*_{2222}, C^*_{2233}, C^*_{3333}, C^*_{1112}, C^*_{2212}, C^*_{3312}$, C^*_{1212} and $C^*_{1313}, C^*_{1323}, C^*_{2323}$ of a fiber reinforced composite with parallelogram periodic cell and spring-like imperfect contact at the interface. These closed–form expressions provide explicit dependence of the overall properties on: (a) The mechanical properties of the constituents, (b) the volumetric fraction of each material phase and, (c) the dimensions

of the representative volume of analysis and (d) the tangential imperfect interface parameters K_n, K_t and K_s.

Effective inhomogeneity – differential approach

To use the differential approach in the form developed by Sevostianov (2007) and Sevostianov and Kachanov (2007), we first find the elastic constants of the *equivalent homogeneous* fiber of the radius equal to the sum of the radius of the core inhomogeneity and the thickness of the interphase layer. This equivalent inhomogeneity has to produce the same effect on the overall elastic moduli as the above described "structured" inclusion.

We first outline the basic logic of the mentioned differential technique first proposed by Shen and Li (2003, 2005) on the base of Mori-Tanaka effective field scheme. We denote by C one of the in-plane elastic moduli, either the bulk one, $k = \left(C_{1111} + C_{1122}\right)/2$, or the shear one, $G = C_{1313}$. The interface layer has the inner and the outer radii R_2 and $R_1 = R_2 + h$, respectively (Fig. 2).

Fig. 2 Fiber and interphase layer for the differential approach.

The inner core of radius R_2 has modulus $C^{(2)}$ and the interface modulus varies across the thickness: $C^I = C^I\left(R\right)$. We aim at finding modulus $C^{eq} = C^{eq}\left(R_1\right)$ of the equivalent homogeneous inclusion of radius R_1. We consider a certain "current" radius $R_2 < r < R_1$ and then add an incremental layer dR of the interface material, $R \rightarrow R + dR$, assuming that the inclusion of radius R is homogeneous (homogenized at the previous step, Fig. 2). To find the corresponding increment of modulus of the equivalent homogeneous inclusion, we model this enlargement by placing the inclusion

of radius R into a matrix that has the property of the interface $C^I(R)$. Volume fraction of the mentioned inclusion in the matrix is the ratio of the volume of the inclusion of radius R to the volume of the enlarged inclusion of radius $R+dR$. It is close to unity and, to the first order, is $1-2dR/R$.

To find the modulus C^{eq} of the equivalent homogeneous inclusion, we treat this system as a composite with volume fraction of inclusions approaching unity and use Hashin lower bound (Hashin, 1965) to estimate its effective properties. This bound is actually accurate in the present context: for an inclusion that is stiffer than the matrix ($k_2 > k_1$, $G_2 > G_1$), of all the cylindrical inclusion shapes of given volume, the circular cross-section produces the smallest effect on the overall elasticity (see, for example Walpole,1969). For in-plane bulk and shear moduli, Hashin lower bounds have the following form:

$$C_{LB} = C^{(1)} + \frac{V_2}{\dfrac{1}{\left(C^{(2)} - C^{(1)}\right)} + \dfrac{\left(1-V_2\right)}{\alpha_C^{(1)} C^{(1)}}} \tag{60}$$

Parameter $\alpha_C^{(1)}$ is expressed in terms of Poisson's ratio ν_I of the interface layer: $\alpha_C^{(1)} = \left(5-4\nu_I\right)/\left[2\left(1+\nu_I\right)\right]$ if C is in-plane bulk modulus, $\alpha_C^{(1)} = \left(5-4\nu_I\right)/\left(4-5\nu_I\right)$ if C is the in-plane shear modulus and $\alpha_C^{(1)} = 2$, if C is out-of-pane shear modulus. Two other transversely-isotropic constants are not independent and are connected with the mentioned three by Hill's conditions

$$\frac{\|C_{1111}\| + \|C_{1122}\|}{2\|C_{1133}\|} = \frac{C_{1111}^{(2)} + C_{1122}^{(2)} - C_{1111}^{V} - C_{1122}^{V}}{2\left(C_{1133}^{(2)} - C_{1133}^{V}\right)} = \frac{C_{1133}^{(2)} - C_{1133}^{V}}{C_{3333}^{(2)} - C_{3333}^{V}} \tag{61}$$

where superscript V denotes the Voigt arithmetic average (for instance, $C_{1133}^{V} = \left(1-V_2\right)C_{1133}^{(1)} + V_2 C_{1133}^{2}$).

Although ν_I may be variable across the interface, its variation does not produce substantial changes in α_C^I. Indeed, changes of ν_I between 0.2 and 0.4 (that is very large, corresponding, for example to variation between ceramics and polymer!) yields variation in α_C^I for in-plane shear modulus between 1.4 and 1.8 respectively. For in-plane bulk modulus – the variation is slightly higher – between 1.75 and 1.21. In absence of any data on this variability, we assume α_C^I to be constant across the interface. To reduce the error produced by such an approximation to several percent, value of ν may be taken as the average between the extreme ones. Applying result (60) to the system "core inclusion – inhomogeneous layer" yields, in the limit $V_2 \to 1$,

$$C^{eq} = C^{(2)} - \left(1 - V_2\right) \frac{\left\|C\right\|^2}{\alpha_C^{(1)} C^{(1)}} \tag{62}$$

Taking into account that $V_2 = 1 - 2\,dR/R$, one arrives at the following nonlinear differential equation for $C^{eq} = C^{eq}\left(R\right)$ of the equivalent homogeneous inclusion:

$$\frac{dC^{eq}\left(R\right)}{dR} = -\frac{2\left[C^{eq}\left(R\right) - C^I\left(R\right)\right]}{\alpha_C^I R}\left[\frac{C^{eq}\left(R\right)}{C^I\left(R\right)} - 1\right] \tag{63}$$

subject to the initial condition $C^{eq}\left(r_0\right) = C^{(2)}$ where $C^{(2)}$ is the modulus of the inner core. Solving this equation and setting $R = R_1 = R_2 + h$ gives $C^{eq}\left(R_1\right)$.

In the simplest case of interest when interface zone is *homogeneous*, $C^I\left(R\right) = \text{const} = C^I$, equation (63) allows the following analytical solution

$$\frac{C^{eq}\left(R_1\right)}{C^{(2)}} = \frac{C^I}{C^{(2)}}\left[\frac{\alpha_C^I}{2\ln\left(BR_1/R_2\right)} + 1\right];\ \text{where}\ B = \exp\left\{\frac{\alpha_C^I C^I}{2\left(C^{(2)} - C^I\right)}\right\} \tag{64}$$

Now, effective properties of a fiber reinforced composite with any arrangement of fibers may be calculated using equation (64) and asymptotic homogenization technique. For

rombic arrangement of fibers, we can use results of formula (2.9) in Rodríguez-Ramos et al. (2012).

Analysis of numerical results

In this section, we compare predictions via analytical expressions (42)-(50), (55) and (59) with results ibtained by differential approach discussed in the previous section. The computations by AHM were made for $N_0 = 10$, where N_0 denotes the number of equations considered in the solution of the infinite algebraic system of equation (38). In general, for low volume fraction of fiber ($V_2 < 0.4$) the accuracy and convergence of the results are good for much smaller values of N_0 ($N_0 \le 2$). More terms are required for high volume fraction of fibers.

Mechanical properties for two different fiber-reinforced composites A and B with isotropic interphases are calculated. The material parameters used in the calculations are given in Table 2.

Fig. 3 illustrates comparison between AHM-spring model and Differential approach (DA) for a two phase composite A with rhombic cell $w_1 = 1,\ \ w_2 = e^{i\theta}, \theta = 75^o$ for two different interphase thicknesses t. Fig. 3 illustrates the normalized effective bulk and transverse shear moduli as functions of the shear modulus of the interphase for different values of ratio $\eta = t/R$. The relation between the spring parameters and the interphase properties are given by relations $K_t = K_s = m_I/\eta,\ \ K_n = E_I(1\text{-}\nu_I)/(1\text{+}\nu_I)(1\text{-}2\nu_I)\eta$ (see Hashin 2002) where the interphase properties are denoted by E_I, m_I and ν_I.

Table 2. Material parameters used in the computations.

The effective elastic moduli were calculated for fiber volume fraction $V_2 = 0.4$. In all the figures there exist an interval ($0 < \log(m_I/m_1) < 5$) where the interphase does not produce noticeable effect on the effective properties. This interval grows up with decreasing thickness of the interphase DA and AHM-spring models show that the effective properties become weaker as the interphase properties $(m_I/m_1) \to 0$. Vice versa, as $(m_I/m_1) \to \infty$ the effective stiffnesses grow up. The thin interphase $\eta = 0.1$ reveals that the differential approach is affected by the thickness of the interphase which characterizes the properties of three-phase composites. As $(m_I/m_1) \to \infty$ and $\eta = 0.001$ both models describe the effective properties of two-phase composite with perfect contact at the interface. In particular the differential approach shows dependence on the contrast of the properties m_2/m_1 and the thickness t. As $m_2/m_1 = 400$ is very large and $\eta = 0.001$ the models coincide in the whole range of interphase properties.

Fig. 3. Comparison between AHM-spring model and differential approach (Diff. Scheme A)

To investigate the consistency of the present model for composites with periodic square cell a comparison with experimental results is shown in Fig. 4. The comparison between AHM spring model for differents values of imperfect parameter $K = K_t = K_n$ and experimental data reported by Hui-Zu and Tsu-Wei (1995) is given. Fig. 4 illustrates the analytical transverse Young's modulus E_t^*/E_1 of the $SCS_6 / Ti-15-3$ composite B. One can observe good agreement between experimental results and analytical predictions for $K = 7,14$. The case $K = 10^{10}$ corresponds to the perfect

bonding and $K = 10^{-10}$ to completely debonded fibers. The corresponding curves are upper and lower bounds respectively for all the curves obtained. These curves coincide with the curves in Fig. 15 in the paper of Hui-Zu and Tsu-Wei (1995).

Fig. 4. Comparison between AHM spring model for differents values of imperfect parameter K and experimental data reported in Fig. 15 by Hui-Zu and Tsu-Wei (1995)

Conclusions

Analytical expressions for effective elastic stiffnesses of a fiber reinforced composite with imperfect contact between matrix and fibers are obtained using asymptotic homogenization method for parallelogram-like arrangement of fibers.

The Hill universal relations are satisfied for these formulae. The following items are obtained from the previous numerical analysis where the interface effects are modeled as distributed mechanical springs.

(a) The mechanical behaviors of fibrous composite are influenced by the stiffness of the spring parameter. The present numerical results of the effective shear and Young modulus versus the fiber volume ratio are correlated very well with other results for perfect and imperfect bonding case, and by the three phase model.

(b) At the fixed fiber volume ratio, the effective shear modulus of composite increases as the stiffness of the interphase increases.

(c) In the asymptotic limit, we can simulate different degrees of the interface's response, the case of the perfect bonding as $K_t, K_s \to \infty$ and complete separation of the matrix and inclusions as $K_t, K_s \to 0$.

(d) The analytical expressions are validated by experimental results.

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Table(s)

${}_{11}L$	${}_{22}L$	${}_{33}L$	${}_{23}L$	${}_{13}L$	${}_{12}L$
C^*_{1111}	C^*_{1122}	C^*_{1133}	0	0	C^*_{1112}
C^*_{2211}	C^*_{2222}	C^*_{2233}	0	0	C^*_{2212}
C^*_{3311}	C^*_{3322}	C^*_{3333}	0	0	C^*_{3312}
0	0	0	C^*_{1313}	C^*_{1323}	0
0	0	0	C^*_{2313}	C^*_{2323}	0
C^*_{1211}	C^*_{1222}	C^*_{1233}	0	0	C^*_{1212}

Table 1. Effective properties related to the local problems

Composites		Young´s modulus E (GPa)	Poisson’s ratio ν	Shear modulus m (GPa)
A	Fiber	960	0.2	400
	Matrix	2.78	0.39	1
B	SCS_6 fiber	399.6	0.250	159.8
	Ti-15-3 alloy	92.3	0.351	34.2

Table 2. Material parameters used in the computations.

Figure(s)

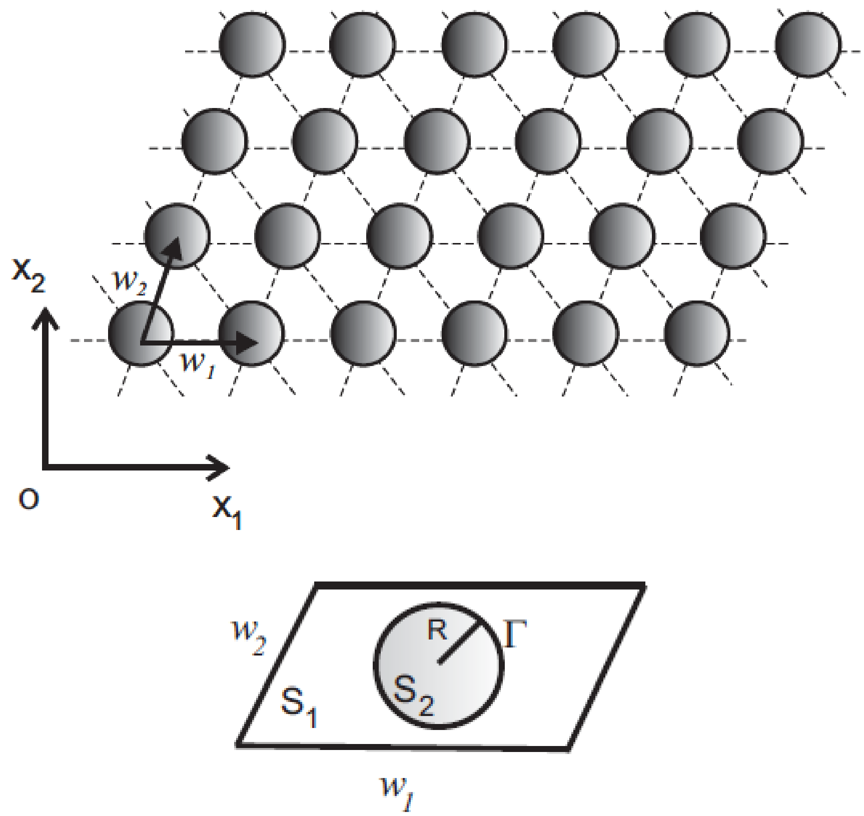

Fig. 1. The heterogeneous medium and the parallelogram periodic cell

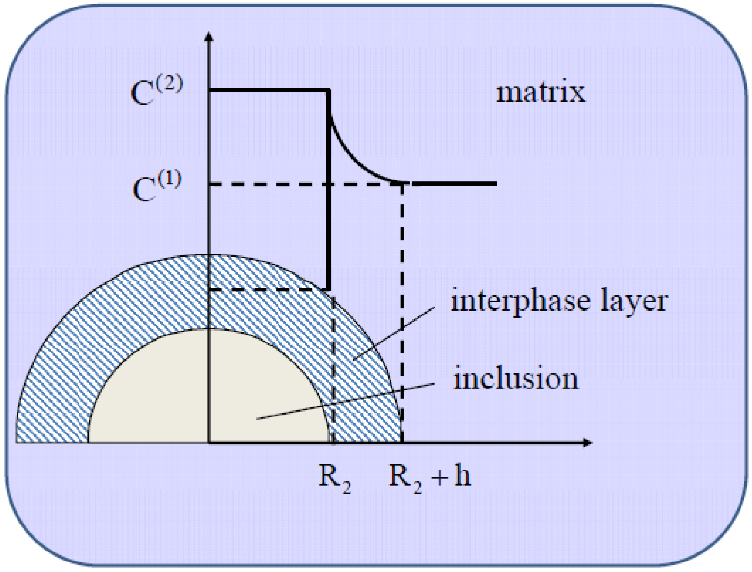

Fig. 2 Fiber and interphase layer for the differential approach.

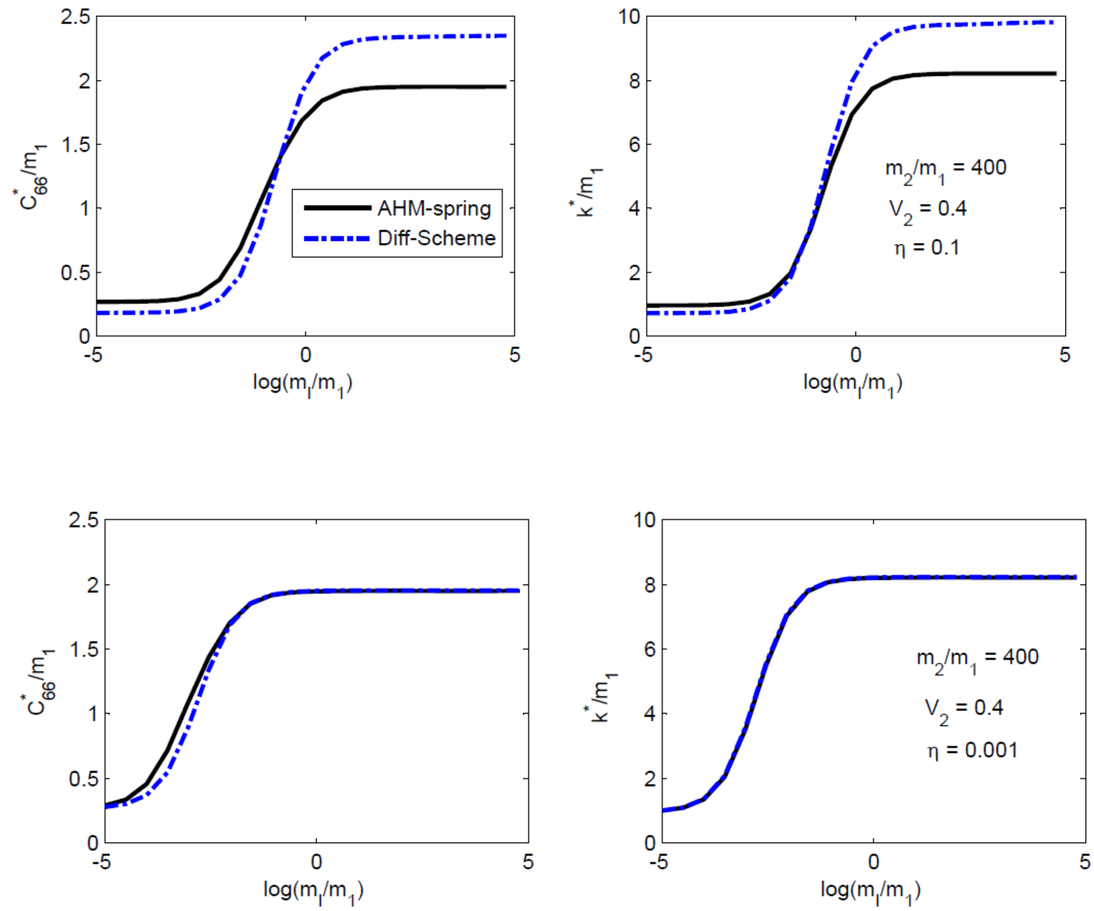

Fig. 3. Comparison between AHM-spring model and differential approach (Diff. Scheme A)

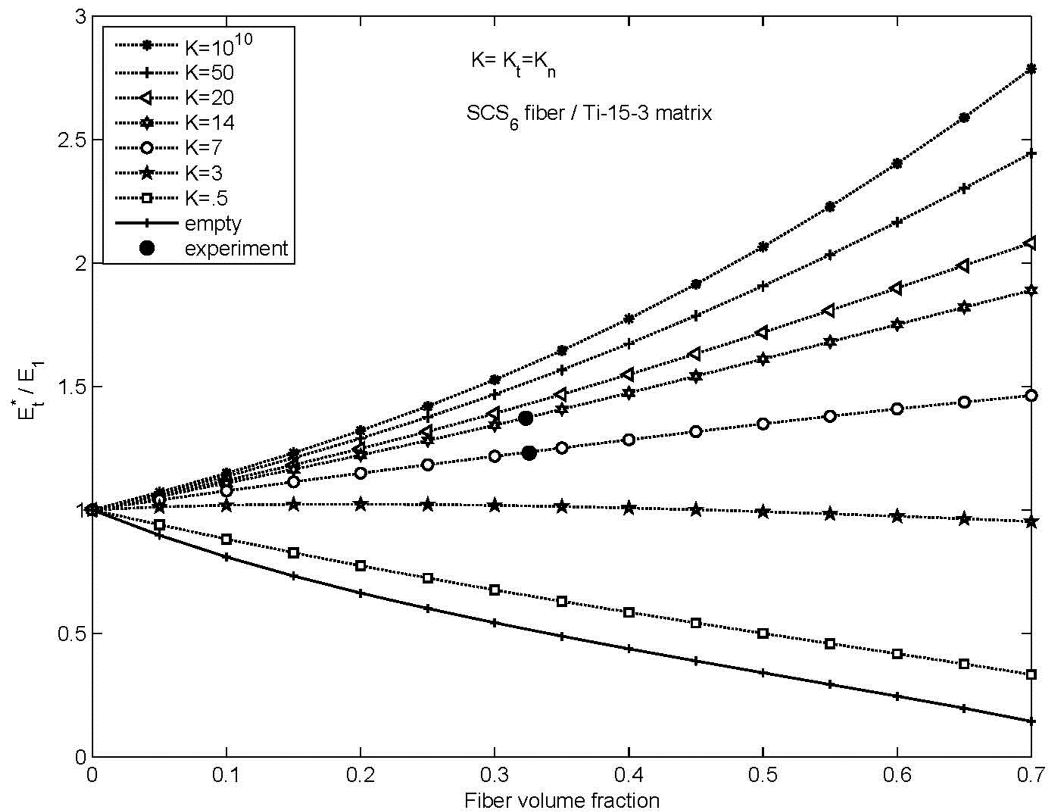

Fig. 4. Comparison between AHM spring model for differents values of imperfect parameter K and experimental data reported in Fig. 15 by Hui-Zu and Tsu-Wei (1995)