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Two analytical models for the study of periodic fibrous elastic composite with different unit cells

Reinaldo Rodríguez-Ramos [a,*], P. Yan [b], Juan C. López-Realpozo [a], Raúl Guinovart-Díaz [a], Julián Bravo-Castillero [a], Federico J. Sabina [c,1], C.P. Jiang [b]

[a] Facultad de Matemática y Computación, Universidad de La Habana, San Lázaro y L, Vedado, Habana 4, CP-10400, Cuba
[b] School of Aeronautic Science and Engineering, Beijing University of Aeronautics and Astronautics, Beijing 100191, China
[c] Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom[2]

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ABSTRACT

In this work, the effective elastic moduli of two-phase fibrous periodic composites are obtained by means of the Asymptotic Homogenization Method (AHM) and eigenfunction expansion-variational method (EEVM), for different types of parallelogram cells. The constituents exhibit transversely isotropic properties. A doubly periodic parallelogram array of cylindrical inclusions under longitudinal shear is considered. The behavior of the shear elastic coefficient for different geometry arrays of the cell related to the angle of the fibers is studied. Some numerical examples and comparisons with other theoretical results demonstrate that both methods (AHM and EEVM) are efficients for the analysis of composites with presence of rhombic cell. The effect of the configuration of the cells on the shear effective property is observed.

1. Introduction

Analytic expressions for shear elastic effective coefficients of fibrous composites with isotropic elastic constituents for square and hexagonal cells under perfect contact condition at the interfaces are calculated in many previous works. In [1–4] analytical expressions for the effective properties are obtained using Asymptotic Homogenization Method (AHM). Exact expressions for the components of the effective stiffness tensor have been obtained in [5] by analytical averaging of the strain and stress fields. The method combines the principle of superposition and technique of complex potentials. Moreover, eigenfunction expansion-variational method (EEVM) is applied in [6] in order to obtained fracture properties of cracked solids. In [7] is presented a very attractive method for the solution of the problem on the shear of a regular fibrous medium underlying which is the exact solution of the Laplace equation in a strip with an infinite number of circular holes. Only single, quite convergent series are used here. Such approach permitted obtaining values of the elastic moduli for different angle of inclination of the cells. Recently, in [8] the effective elastic moduli of two-phase fibrous periodic composites are obtained for different types of parallelogram cells based on the AHM and making use of potential methods of a complex variable and properties of elliptic and related functions. The constituents exhibit transversely isotropic properties. A doubly period-parallelogram array of cylindrical inclusions under longitudinal shear is considered. The behavior of the shear elastic effective coefficient for different geometry arrays of the cell related to the angle of the fibers is studied.

In this work, micromechanical analysis methods are applied to unidirectional fibers composites with different cross angles of the cells to determine the homogenized elastic properties of a composite. In particular, this work compile the presentation of AHM and EEVM reported by Guinovart-Díaz et al. [8] and Yan et al. [9], respectively to the research of the shear elastic effective coefficient for different geometry arrays of the cell in two-phase fibrous periodic composites. Since the theoretical fundamentals of both methods are in details developed in a separately form, a comparison between the results applied to the calculation of the shear elastic effective coefficient derived for both methods is convenient and it is presented here. Moreover, the difference of the present work with respect to the results reported in [10] consists in the computation of the shear elastic effective properties for different parallelogram one-directional fiber distribution where the symmetry lines define a parallelepiped unit cell, representing the periodic microstructure of a one-directional fiber composite. The effect of the

* Corresponding author.
E-mail addresses: reinaldo@matcom.uh.cu (R. Rodríguez-Ramos), yanpeng117@gmail.com (P. Yan), jclrealpozo@matcom.uh.cu (J.C. López-Realpozo), guino@matcom.uh.cu (R. Guinovart-Díaz), jbravo@matcom.uh.cu (J. Bravo-Castillero), fjs@mym.iimas.unam.mx (F.J. Sabina), jiangchiping@buaa.edu.cn (C.P. Jiang).
1 Address: Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México, Apartado Postal 20-726, Delegación de Álvaro Obregón, 01000 México, D.F., Mexico.
2 Sabbatical leave.

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geometry distributions and the angle of the fibers in the composite are analyzed. The results in this paper are mainly based on the impact of the fiber cross angles on the stiffness properties of the chosen composites. The accuracy of the results from both micromechanics modelling procedure has been compared.

2. Heterogeneous elastic composite: general considerations

Consider elastic materials that respond linearly to changes in the mechanical stress and strain tensors. A two-phase periodic composite is considered which consists of a parallelogram array of identical circular cylinders embedded in a homogeneous medium. The cylinders are infinitely long. The material properties of each phase belong to the crystal symmetry class 6 mm, where the axes of material and geometric symmetry are parallel.

As shown in Fig. 1a, the infinitely extended doubly-periodic structure is obtained from a primitive cell which is repeated in the two directions, where w_1 and w_2 denote the two fundamental periods. The general period P_{nm} can be defined as $P_{nm} = nw_1 + mw_2$, where n and m are arbitrary integers. The displacement fields $\mathbf{u}(\mathbf{x})$ are quasi-periodic and the stress fields $\boldsymbol{\sigma}(\mathbf{x})$ are periodic. Two distinct phases, occupying S_1 (matrix) and S_2 (fiber) (see Figs. 1 and 2) are assumed to be in perfect contact along the interface of each cylinder which is denoted by Γ.

As the body forces are absent, stress, strain and displacement fields satisfy the following three equations, respectively:

$$\text{Stress—strain relations} : \boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}. \tag{1}$$

$$\text{Displacement—strain relations} : \boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \mathbf{u}\nabla). \tag{2}$$

$$\text{Equilibrium equations} : \nabla \cdot \boldsymbol{\sigma} = 0. \tag{3}$$

where $\mathbf{C}$ is the elasticity tensor and ∇ is the gradient operator.

3. Asymptotic Homogenization Method (AHM): antiplane elastic problem

Using AHM the constitutive relations (1) with rapidly oscillating material coefficients are transformed in new physical relations with constant coefficients $\mathbf{C}^e$, which represent the elastic properties of an equivalent homogeneous medium and are called the effective coefficients of composite. The governing elastic Eq. (3) for this kind of materials are the Navier equations of linear elasticity for the mechanical displacement $\mathbf{u} = (u_1, u_2, u_3)$. In a two-dimensional situation, like in the considered geometry here, it turns out that the above equations uncouple into two independent systems under suitable boundary conditions. Just like, the familiar plane and antiplane-strain deformation states in linear elasticity, see [1–4]. One of them involves u_1; u_2, i.e., it is a state of in-plane mechanical deformation fields. The other state, which is of particular interest in this work, is characterized by an out-of-plane mechanical displacement u_3.

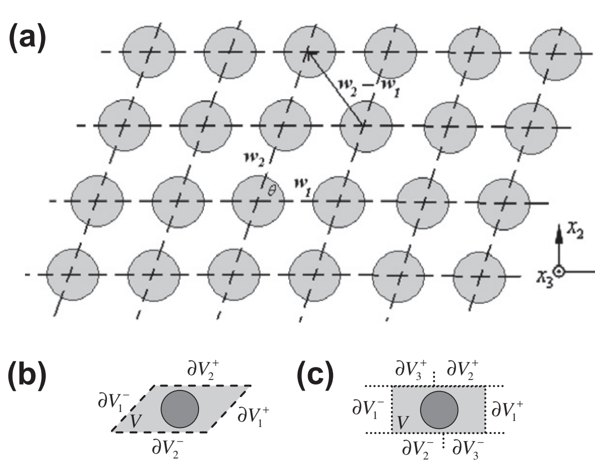

Fig. 1. (a) The cross-sectional view of one-directional fibers at cross angles of θ embedded in matrix. (b and c) two different unit cells with boundaries ∂V_j^+ and ∂V_j^- in pairs ($j = 1, 2$ in (b) and $j = 1, 2, 3$ in (c)) used in EEVM model.

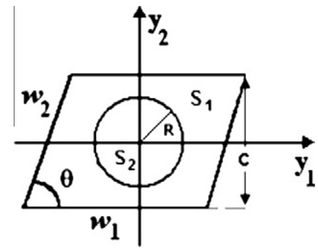

Fig. 2. Parallelogram cell of the composite for different phases used in AHM model.

The aim consists to determine the effective properties using the homogenization method, say, as in [11], considering mechanical perfect conditions at the interfaces. Thus it is only necessary to deal with the component of the displacement u_3.

Let l be the distance between the centers of two neighboring cylinders and L the diameter of the composite. Then, when $\varepsilon = l/L$ is a very small number, it is possible to distinguish two spatial scales, one is x, the slow variable, and the other is $y = x/\varepsilon$, the fast variable. Substituting matrix and fiber constitutive relations (1) into Navier equations (3), inhomogeneous governing equations arise for the herein studied composite (Fig. 1a) that can be solved asymptotically posing the ansatz:

$$u_3(x) = \tilde{u}_0(x, y) + \varepsilon\tilde{u}_1(x, y) + O(\varepsilon^2), \tag{4}$$

and using the method of two scales. The functions $\tilde{u}_i$ ($i = 0, 1$) are found to satisfy certain differential equations related to the original system in a periodic cell. It is a well-known derivation whose details can be found elsewhere and here is omitted. Of a greater interest are the so-called local (or canonical) problems associated here with the correction term $\tilde{u}_1$ to the mean variations $\tilde{u}_0$, since they appear in the formulae of the effective properties. There are two of such problems, which are referred as $_{13}L$ and $_{23}L$. A pre-index is used to distinguish similar constants and functions such as displacements and potentials, which appear below. Due to the linearity of the Eqs. (1)–(3), the correction term $\tilde{u}_1$ can be obtained as a linear combination of some of such displacements and potentials. This, however, will not be done here, since the main objective of this paper is the characterization of the three effective properties C_{1313}, $C_{1323} = C_{2313}$ and C_{2323}. Due to composite and constitutive symmetries, there are one alternative for obtain C_{1313} and C_{2323} and two alternatives for obtain $C_{2313} = C_{1323}$ property as follows,

$$C_{55}^e = C_{44}^{(1)}V_1 + C_{44}^{(2)}V_2 + \langle C_{44}^{(1)}\ {}_{13}N_{,1}\rangle, \tag{5}$$

$$C_{45}^e = C_{54}^e = \langle C_{55}^{(1)}\,{}_{13}N_{,2}\rangle = \langle C_{44}^{(1)}\ {}_{23}N_{,1}\rangle, \tag{6}$$

$$C_{44}^e = C_{44}^{(1)}V_1 + C_{44}^{(2)}V_2 + \langle C_{44}^{(1)}\ {}_{23}N_{,2}\rangle, \tag{7}$$

where the abbreviate index notation $C_{55} = C_{1313}$, $C_{45} = C_{2313}$, $C_{54} = C_{1323}$ and $C_{44} = C_{2323}$ for the elastic properties is used for simplicity. $_{13}N$ and $_{23}N$ are functions of $z = y_1 + iy_2$ that are solutions of the local problems ([1–2]) $_{13}L$ and $_{23}L$, respectively, V_1 and V_2 are the percentages of concentrations or matrix and fiber in the composite, $V_1 + V_2 = 1$, whereas the indices (1) and (2) denotes the matrix and the fiber properties, respectively. The angular brackets

define the volume average per unit length over the area V of the cell, that is, $\langle F\rangle = \frac{1}{V}\int_V F(y)dy$.

3.1. The antiplane problems $_{\alpha 3}L$

The dimensionless antiplane problems $_{\alpha 3}L$ ($\alpha = 1, 2$) over de periodic cell Y are written below, to find doubly periodic functions $u^{(\gamma)} = N^{(\gamma)}/l$ such that

$$\nabla^2 u^{(\gamma)} = 0, \text{ in } S_\gamma, \tag{8}$$

$$\|u\| = 0, \text{ on } \Gamma, \tag{9}$$

$$\left\|\frac{C_{44}(u_{,1}n_1 + u_{,2}n_2)}{C_{44}^{(1)}}\right\| = -(1-\rho)n_\alpha, \text{ on } \Gamma, \tag{10}$$

where $\rho = C_{44}^{(2)}/C_{44}^{(1)}$, the comma notation denotes a partial derivative relative to the ξ_δ with $\xi = y/l$, S_γ ($\gamma = 1, 2$) is the region occupied by the phases (Fig. 2). Moreover, ∇^2 is Laplace's operator. The outward unit normal vector to the interface Γ is $\mathbf{n} = (n_1, n_2)$. The double bar notation $\|f\|$ denotes the jump of the function f across the interface Γ. The well-developed theory of analytical functions (see, [12]) can be applied to solve this problem, as in [11]. The effective coefficients C_{55}^e, C_{45}^e, C_{44}^e and C_{54}^e in (5)–(7) are connected by the following relations

$$C_{55}^e + iC_{45}^e = C_{44}^{(1)} - \frac{2V_2}{R}a_1, \tag{11}$$

$$C_{54}^e + iC_{44}^e = iC_{44}^{(1)} - \frac{2V_2}{R}a_1, \tag{12}$$

where $a_1 = \chi R(1,i)(I + \chi R^2 J - \chi^2 N_1(I+\chi W)^{-1}N_2)^{-1}B$, I is the unit matrix,

$$J = \begin{pmatrix} h_{11}+h_{12} & h_{21}-h_{22} \\ -h_{21}-h_{22} & h_{11}-h_{12} \end{pmatrix},$$

with $h_{1\gamma} = \mathrm{Re}\{H_\gamma\}$, $h_{2\gamma} = \mathrm{Im}\{H_\gamma\}$, $H_1 = \frac{\bar{\delta}_1\bar{w}_2 - \bar{\delta}_2\bar{w}_1}{w_1\bar{w}_2 - w_2\bar{w}_1}$, $H_2 = \frac{\delta_1\bar{w}_2 - \delta_2\bar{w}_1}{w_1\bar{w}_2 - w_2\bar{w}_1}$, $\delta_\gamma = \zeta(z+w_\gamma) - \zeta(z)$, $\zeta(z)$ is the Weierstrass' ζ quasi periodic function. The matrix W is composed of square blocks of order 2, defined by

$$w_{kp} = \begin{pmatrix} w_{1kp} & -w_{2kp} \\ -w_{2kp} & -w_{1kp} \end{pmatrix},$$

where $w_{1kp} = Re\{w_{kp}\}$, $w_{2kp} = \mathrm{Im}\{w_{kp}\}$, $N_1 = W(w_{k1})$, $N_2 = W(w_{1p})$, $w_{kp} = \frac{(k+p-1)!}{(k-1)!(p-1)!}\frac{S_{k+p}R^{k+p}}{\sqrt{k}\sqrt{p}}$, $S_{k+p} = \sum'_{m,n}(mw_1 + nw_2)^{-(k+p)}$, the apostrophe in the summation symbol means that the pair $(m, n) = (0.0)$ is excluded, $k + p > 2$, $S_2 = 0$, $\chi = \frac{1-\rho}{1+\rho}$. The transpose of infinite vectors B have the form $B^T = (\delta_{1\alpha}, \delta_{2\alpha}, 0, 0, \ldots)$, δ_{kp} is the δ-Kronecker. $V_2 = \pi R^2/V$ and $V = |w_1||w_2| \sin\theta$.

4. Eigenfunction expansion-variational method (EEVM)

Recently [6,9] presented a new variational functional for a unit cell of a heterogeneous solid with periodic microstructures by incorporating the quasi-periodicity of the displacement field and the periodicity of the stress and strain fields into the strain energy functional. The functional can accommodate a broad class of periodic structure including the case where symmetry or antisymmetry properties of the unit cell may not exist. By combining with the eigenfunction expansions of the complex potentials satisfying the continuity conditions on the fiber–matrix interface, an eigenfunction expansion-variational method (EEVM) based on a unit cell is developed, which is also used to deal with fibrous elastic composite with general oblique cells for a comparison. For the periodic microstructure Fig 1a, the unit cell may be different in shape and two different unit cells are shown in Fig. 1b and c. From the doubly periodic distribution of unit cells, the boundary of a unit cell can be divided into ∂V_j^+ and ∂V_j^- in pairs ($j = 1, 2$ in Fig. 1b and $j = 1, 2, 3$ in Fig. 1c). The loading condition of the composite can be prescribed by an average strain over the unit cell, $\langle\varepsilon\rangle$, which is the same for every unit cell. The corresponding displacements and tractions ($\mathbf{t} = \boldsymbol{\sigma}\cdot\mathbf{n}$) on ∂V_k^+ and ∂V_k^- satisfy the following periodic boundary conditions:

$$\mathbf{u}^{k+} - \mathbf{u}^{k-} = \langle\boldsymbol{\varepsilon}\rangle\cdot\mathbf{p}^k \tag{13}$$

$$\mathbf{t}^{k+} + \mathbf{t}^{k-} = 0 \tag{14}$$

where $p^1 = w_1$ and $p^2 = w_2$ for Fig. 1b and $p^1 = w_1$, $p^2 = w_2$ and $p^3 = w_2 - w_1$ for Fig. 1c whereas $\mathbf{u}^{k+} - \mathbf{u}^{k-}$ denotes the displacement difference between the opposite boundaries (see ∂V_k^+ and ∂V_k^- in Fig. 1b and c) of a unit cell.

Presupposing the satisfaction of the stress–strain relations, the displacement–strain relations and fiber–matrix interface conditions and absorbing the periodic boundary conditions of unit cells (13) and (14) by using the Lagrangian multiplier method, a generalized potential energy functional can be defined as:

$$\Pi = \int_V \frac{1}{2}\boldsymbol{\varepsilon} : \mathbf{C} : \boldsymbol{\varepsilon}\, dV - \sum_k \int_{\partial V_k^+} \mathbf{t}^{k+}\cdot(\mathbf{u}^{k+} - \mathbf{u}^{k-} - \langle\varepsilon\rangle\cdot\mathbf{p}^k)dS, \tag{15}$$

where the second term is corresponding to the displacement coupling conditions derived from the quasi-periodicity of the displacement field. The functional (15) is suitable for a general unit cell with any microstructure, where symmetry and antisymmetry properties of the unit cell may not exist.

If the stress fields satisfy the equilibrium equations in addition, only the periodic boundary conditions of the unit cell must be satisfied, the stationary conditions for the functional can written as

$$\sum_k \int_{\partial V_k^+} \delta\mathbf{t}^{k+}\cdot(\mathbf{u}^{k+} - \mathbf{u}^{k-})dS - \sum_k \int_{\partial V_k^+} (\mathbf{t}^{k+} + \mathbf{t}^{k-})\cdot\delta\mathbf{u}^{k-}\, dS$$
$$= \sum_k \int_{\partial V_k^+} \delta\mathbf{t}^{k+}\cdot\langle\varepsilon\rangle\cdot\mathbf{p}^k\, dS, \tag{16}$$

where $\delta(\cdot)$ denotes the variation.

For the longitudinal shear problem (antiplane problem), the displacement field $u_3 \equiv u$ satisfies Laplace's equation $\nabla^2 u = 0$. Referring to Muskhelishvili [12] the displacement u and the stresses σ_{13} and σ_{23} can be expressed by a complex potential (an analytical function) $f(z)$:

$$u = \frac{1}{2}[f(z) + \overline{f(z)}] \tag{17}$$

$$\sigma_{13} - \mathrm{i}\sigma_{23} = C_{44}f'(z) \tag{18}$$

where the prime denotes the derivative with respect to z.

Let the subscripts 1 and 2 refer to the matrix and fiber in a unit cell and the origin of the complex z-plane be at the center of the fiber. Then both the complex potentials $f_1(z)$ in the matrix and $f_2(z)$ in the fiber can be expanded into power series, which include infinite positive and negative power terms. Since the displacement in the fiber is finite, the negative power terms of $f_2(z)$, which lead to an infinite displacement at the origin, do not appear. On the other hand, the origin is outside the matrix, the negative power terms of $f_1(z)$ do not lead to an infinite displacement in the matrix, and are retained. Hence $f_1(z)$ in the matrix domain can be expanded into a Laurent series and $f_2(z)$ in the fiber domain a Taylor series:

$$f_1(z) = \sum_{n=1}^{\infty} G_n z^{-(2n-1)} + \sum_{n=1}^{\infty} F_n z^{2n-1} \tag{19}$$

$$f_2(z) = \sum_{n=1}^{\infty} E_n z^{2n-1} \tag{20}$$

where E_n, G_n and F_n are complex coefficients.

The interfacial conditions at the perfect fiber–matrix interface are used to determine the relations of the complex coefficients:

$$E_n = (C_{44}^{(1)}/C_{44}^{(2)})(F_n - \bar{G}_n R^{2-4n}), \quad \bar{G}_n = \frac{C_{44}^{(1)} - C_{44}^{(2)}}{C_{44}^{(1)} + C_{44}^{(2)}} R^{4n-2} F_n \tag{21}$$

Then the remaining coefficients can be determined by using the stationary condition (Eq. (16)). The displacement and stress fields are obtained from Eqs. (17) to (18). The effective properties of the composite C^e can be determined with the aid of the average field theory:

$$\langle \boldsymbol{\sigma} \rangle = C^e : \langle \boldsymbol{\varepsilon} \rangle \tag{22}$$

where $\langle \boldsymbol{\sigma} \rangle$ and $\langle \varepsilon \rangle$ are the average stress and strain tensors in a unit cell, respectively.

The package MATHEMATICA is used to implement the computations. A finite number of expansion terms are taken to achieve convergent results, and the number is generally less than ten except for some extreme cases.

5. Analysis of the results

(1) The variation of effective properties depending on fiber volume fraction is examined. Fig. 3 shows a comparison for the effective coefficients $C_{55}^e/C_{44}^{(1)}, C_{45}^e/C_{44}^{(1)}, C_{44}^e/C_{44}^{(1)}$ between the results obtained by AHM, using the formulae (11), (12) and by EEVM, using the formula (19), with angle of $\theta = 30^0$ for the periodic cell. AHM (N_0 = 1 or 4) denotes that calculations were made taking matrices of order N_0 in (11) and (12). Very good match between both approaches can be appreciated. AHM (N_0 = 1) provides a good estimate of the effective properties of volume fraction below to 0.3 and holds the information of the orthotropic composite.

(2) Thermal conductivity is a very important property for the applications of fiber reinforced composites. The present model is very useful for estimating the conductivity properties of fibers composites. To investigate the consistency of the calculation with experiments results, Figs. 4 and 5 show a comparison between AHM and EEVM, for hexagonal and square periodic cell with the approach in [13] and experimental data reported by Thornburg and Pears [14] which can be indirectly taken from Fig. 4 by [13]. In Fig. 4 we considered soft inclusion with ratio ρ = 4.4 fibers/matrix and in Fig. 5 rigid inclusion with ratio ρ = 666. From the Figs. 4 and 5 a great overlap between the different models can be appreciated. AHM-hex and EEMV-hex are coincident and the first approximation AHM-(N_0 = 1) are provides very good estimation of the effective properties. AHM for the first approximation (N_0 = 1) does not establish any difference between square and hexagonal periodic cell. It coincides with Mori–Tanaka's model.

(3) The dependence of the values of the elastic coefficients $C_{55}^e/C_{44}^{(1)}, C_{45}^e/C_{44}^{(1)}, C_{44}^e/C_{44}^{(1)}$ for given ratios between the shear modulus $\rho = C_{44}^{(2)}/C_{44}^{(1)}$, the magnitude of the volume fraction V_2 and the angle of the periodic cell is considered. Different configurations of cells are studied, i.e. rhombic, parallelogram and rectangular periodic cells. A comparison between

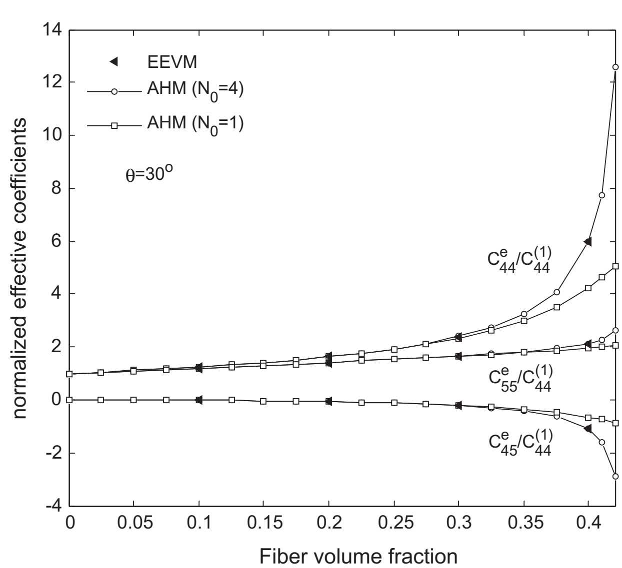

Fig. 3. Comparison of two order of aproximation in the formulae (11) and (12) of model AHM with EEVM model, for a composite with rombic periodic cell of angle θ = 30°.

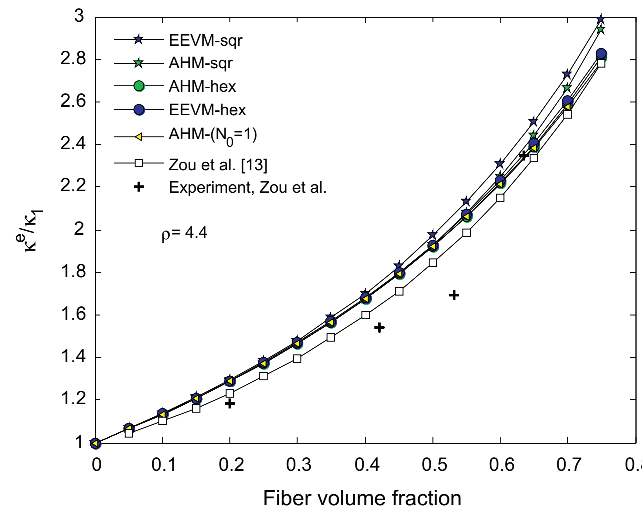

Fig. 4. Comparison between the results obtained for hexagonal and square array by AHM, EEVM of the present work and theoretical and experimental results reported in [14] for thermal conductivity κ and ratio ρ = 4.4 fibers/matrix.

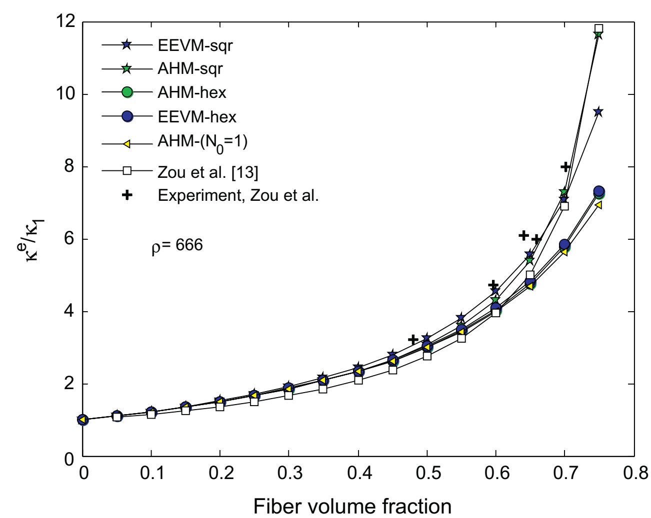

Fig. 5. Comparison between the results obtained for hexagonal and square array by AHM, EEVM of the present work and theoretical and experimental results reported in [14] for thermal conductivity κ and ratio ρ = 666 fibers/matrix.

Table 1
Comparison between the results reported by [7] (G&N), EEVM and AHM for a rhombic periodic cell, with different volume fractions and ratios $\rho = C_{44}^{(2)}/C_{44}^{(1)}$.

$\frac{C_{44}^{(2)}}{C_{44}^{(1)}}$	V_2	$C_{55}^e/C_{44}^{(1)}$			$C_{45}^e/C_{44}^{(1)}$			$C_{44}^e/C_{44}^{(1)}$		
		G&N	AHM	EEVM	G&N	AHM	EEVM	G&N	AHM	EEVM
20	0.3	1.75	1.75	1.75	0.02	0.02	0.02	1.74	1.74	1.74
	0.5	2.70	2.70	2.70	0.08	0.08	0.08	2.66	2.66	2.66
	0.7	5.00	5.00	5.00	0.34	0.34	0.34	4.83	4.83	4.83
120	0.3	1.84	1.84	1.84	0.02	0.02	0.02	1.83	1.83	1.83
	0.5	3.01	3.01	3.01	0.11	0.11	0.11	2.96	2.96	2.96
	0.7	6.47	6.47	6.47	0.60	0.60	0.60	6.16	6.16	6.16

Table 2
Comparison between the results reported by [7] (G&N), EEVM and AHM for different parallelogram periodic cells, different volume fractions and ratio $\rho = 120$.

$\|w_2\|$	V_2	$C_{55}^e/C_{44}^{(1)}$			$C_{45}^e/C_{44}^{(1)}$			$C_{44}^e/C_{44}^{(1)}$		
		G&N	AHM	EEVM	G&N	AHM	EEVM	G&N	AHM	EEVM
1.10	0.3	1.88	1.88	1.88	0.01	0.01	0.01	1.8	1.79	1.79
	0.5	3.27	3.27	3.27	0.06	0.06	0.06	2.77	2.77	2.77
	0.6	4.88	4.88	4.88	0.13	0.13	0.13	3.59	3.59	3.59
1.25	0.3	1.95	1.95	1.95	0.01	0.01	0.01	1.75	1.75	1.75
	0.5	3.86	3.86	3.86	0.03	0.03	0.03	2.56	2.56	2.56
	0.6	7.9	7.92	7.92	0.05	0.05	0.05	3.18	3.18	3.18
1.4	0.3	2.04	2.04	2.04	0.00	0.002	0.002	1.71	1.71	1.71
	0.5	5.05	5.05	5.05	0.01	0.01	0.01	2.41	2.41	2.41

Table 3
Comparison between the results reported by [7] (G&N), EEVM and AHM for different rectangle periodic cells, different volume fractions and ratio $\rho = 120$.

$\|w_2\|$	V_2	$C_{55}^e/C_{44}^{(1)}$			$C_{44}^e/C_{44}^{(1)}$		
		G&N	AHM	EEVM	G&N	AHM	EEVM
1.10	0.3	1.89	1.89	1.89	1.80	1.80	1.80
	0.5	3.31	3.31	3.31	2.79	2.79	2.79
	0.6	5.10	5.11	5.11	3.65	3.65	3.65
1.25	0.3	1.96	1.96	1.96	1.75	1.75	1.75
	0.5	3.98	3.98	3.98	2.56	2.56	2.56
	0.6	9.17	9.23	9.23	3.18	3.18	3.18
1.4	0.3	2.05	2.05	2.05	1.71	1.71	1.71
	0.5	5.33	5.33	5.33	2.40	2.40	2.40

the results reported by [7] (G&N), EEVM and AHM both reported in the present work is given in the Tables 1–3, respectively. Notice good accuracy of the calculations using AHM and EEVM model with [7] (G&N) approach. The tables reported in [7] provided us likelihood of comparison and estimation of the physical parameters. In Table 1 of comparisons is related to rhombic periodic cell with period $|w_1| = |w_2| = 1$, θ = arcos (1/4), $\rho = 20$ and $\rho = 120$. Table 2 is related to parallelogram periodic cells with $w_1 = 1$, θ = arcos $(0.25|w_2|)$ and $\rho = 120$. Table 3 is related to rectangle periodic cell with $w_1 = 1$ and ratio $\rho = 120$.

6. Conclusions

The two-dimensional antiplane problem for two phase elastic composite with a doubly-periodic array of cells of circular cross-section fibers is studied. The numerical analysis shows a good convergence of both presented method, i.e. AHM and EEVM, even though the fiber volume fraction is relatively high. Comparisons with the existing analytical and experimental results demonstrate the accuracy and validity of the approximation formula for different type of cells, for instance, rhombic, parallelogram and rectangular periodic cells. Besides, the models confirm the strong dependence of the effective properties related to the cells.

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