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Influence of parallelogram cells in the axial behaviour of fibrous composite

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Effective longitudinal shear moduli closed-form analytical expressions of two-phase fibrous periodic composites are obtained by means of the asymptotic homogenization method (AHM) for a parallelogram array of circular cylinders. This work is an extension of previous reported results, where elastic, piezoelectric and magneto-electro-elastic composites for square and hexagonal arrays with perfect contact were considered. The constituents exhibit transversely isotropic properties. A doubly period-parallelogram array of cylindrical inclusions under longitudinal shear is studied. The behaviour of the anisotropic shear elastic coefficients is studied for several cell geometry arrays. Numerical examples and comparisons with other theoretical results demonstrate that the present model is efficient for the analysis of composites in which the periodic cell is rectangular, rhombic or a parallelogram. The effect of the arrangement of the cells on the shear effective property is discussed. The present method can provide benchmark results for other numerical and approximate methods.

1. Introduction

Theoretical predictions of effective properties as well as the micro-mechanical simulation of composite materials are important problems in science and engineering. They have received considerable attention of many authors since the pioneering studies of [1–2]. The issue has been addressed in a number of publications for both experimental and numerical estimation of effective properties for this kind of composite. The topic remains of interest for several reasons: (i) different effective properties which can be determined in entirely the same mathematical way, elastic moduli, electrical conductivity, thermal conductivity, etc. A list of different applications of the effective properties was summarized by Hashin in [3] and Andrianov et al. in [4]; (ii) this problem is the basis for understanding complex microstructures relationships in the composite and move towards the study of composite with more complex links between the constituents, for example, the study of effective properties of piezoelectric or magneto-electro-elastic composites as in [5–7]; (iii) composites are connected with new areas of research, for example, in [8–9], a new model of cortical bone elasticity is developed and used to assess the influence of mesoscale porosity on the induced anisotropy of the material. Bisegna and Caselli in [10] present a mathematical model in order to study the effective complex conductivity of a biological tissue comprising tubular cells; (iv) with the

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advance of technology in the production of composite is possible to present more realistic mathematical models that can be used in practice.

The problem of the shear of a medium with congruent groups of fibers was solved in [11–13] by using doubly periodic functions. The displacement within cell limits is represented by Fourier series in [14–15] and the method of least squares is used to satisfy the conditions on the boundary. Some other authors have proposed other different models for studying physical properties in composites, small fraction of these results are detailed in [16–27]. Many of these known methods are able to provide accurate numerical results only in certain partial cases of the problem. Recently, Kushch and co-workers in [28] developed an analytical tool for computing the effective properties of unidirectional fiber composite possessing arbitrary anisotropy; Jiang et al. [29] analyzed different situations of rhombic composite.

In this work, micro-mechanical analysis method is applied to unidirectional fibers composites with different period-parallelogram cell to determine the homogenized elastic properties of a composite. Analytical expressions of the elastic effective coefficients for different arrays of fibrous composites with transversely isotropic electro-elastic constituents are calculated using the AHM reported in [30–33]. The difference of the present work with respect to the results reported in [13,30] consists in the computation of the shear elastic effective properties for different one-directional fiber distribution where the symmetry lines define a parallelepiped unit cell, representing the periodic microstructure of the composite. Moreover, the effect of the spatial distributions, where the periodic cells can be either rectangular, rhombic or parallelogram, are analyzed. The results, in this work, are mainly focused in the estimation of analytical expressions for the effective coefficients taking into account the impact of the different periodic cells over the stiffness properties. The accuracy of the present model has been compared with other theoretical models and experimental data.

2. General considerations

A two-phase periodic composite is considered in this work which consists of a parallelogram array of identical circular cylinders embedded in a homogeneous medium. The cylinders are infinitely long. Figs. 1 and 2 show the composite and primitive cell in the plane normal to cylindrical axis. The material properties of each phase belong to the crystal symmetry class 6 mm, where the axes of material and geometric symmetry are parallel. The governing elastic equations for this kind of materials are the Navier equations of linear elasticity for the mechanical displacement $\mathbf{u} = (u_1, u_2, u_3)$. In a two-dimensional situation, like in the special distribution considered herein, it turns out that the above equations uncouple into two independent systems under suitable boundary conditions. Just like the familiar plane- and anti-plane-strain deformation states in linear elasticity, see [12,26,30–33].

One of them involves u_1 and u_2, i.e., it is a state of in-plane mechanical deformation fields. The other state, which is of particular interest in this work, is characterized by an out-of-plane mechanical displacement u_3. The main aim of this paper is the determination of effective properties for the out-of-plane state using the homogenization method, say, as in Camacho-Montes et al. [5], López-López et al. [26], considering mechanical perfect conditions at the interfaces, i.e. continuity of displacements and tractions at the interface. Thus, it is only necessary to deal with u_3. In this case the relevant constitutive relations are

$$\sigma_{13} = 2p\varepsilon_{13}, \quad (1)$$

$$\sigma_{23} = 2p\varepsilon_{23}, \quad (2)$$

where σ_{13}, σ_{23} are the components of out-of-plane mechanical stress, $p = C_{1313} = C_{2323}$ are the elastic modulus, $2\varepsilon_{13} = u_{3,1}$, $2\varepsilon_{23} = u_{3,2}$ denote the components of the mechanical strain and the comma notation is understood to denote differentiation with respect to x_i.

Two distinct phases, occupying S_1 and S_2 (Fig. 2) are assumed to be in perfect contact along the interface of each cylinder which is denoted by Γ

$$\|u_3\| = 0, \quad \|\sigma \cdot n\| = 0. \quad (3)$$

The double bar is used to denote the jump of the relevant function across the interphase taken from the matrix to the fiber. By definition: $\|f\| = f_1 - f_2$ where the subscripts 1 and 2 indicate the matrix and fiber, respectively.

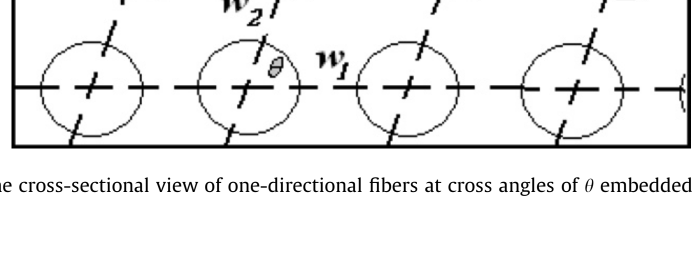

Fig. 1. The cross-sectional view of one-directional fibers at cross angles of θ embedded in matrix.

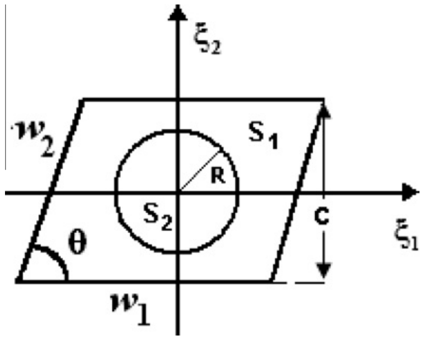

Fig. 2. Parallelogram cell of the composite.

Let l be the distance between the centers of two neighboring cylinders and L the diameter of the composite. Then, when $\varepsilon = l/L$ is a very small number, it is possible to distinguish two spatial scales, one is $\mathbf{x}$, the slow variable, and the other is $\mathbf{y} = \mathbf{x}/\varepsilon$, the fast variable. Substituting matrix and fiber constitutive relations (1)–(3) into Navier equations, inhomogeneous governing equations arise for the herein studied composite (Fig. 1) that can be solved asymptotically posing the ansatz:

$$u_3(\mathbf{x}) = v_0(\mathbf{x}, \mathbf{y}) + \varepsilon v_1(\mathbf{x}, \mathbf{y}) + O(\varepsilon^2), \tag{4}$$

and using the method of two scales. The functions v_0, v_1 are found to satisfy certain differential equations related to the original system in a periodic cell (Fig. 2). It is a well-known derivation whose details can be found in Ref. [12] and here is omitted. Of a greater interest are the so-called local (or canonical) problems associated with the correction term v_1 to the mean variations v_0, since they appear in the formulae of the effective properties. There are two of such problems, which are referred as ${}_{13}L$ and ${}_{23}L$. A pre-index is used to distinguish the functions such as displacements for different local problems, which appear below.

The main objective of this paper is the characterization of the three effective properties: $p_{11} = C_{1313}$, $p_{12} = C_{1323}$, $p_{21} = C_{2313}$ and $p_{22} = C_{2323}$. The composite and constitutive symmetries lead us to find one way for obtaining p_{11} and p_{22} and two alternatives ways for $p_{12} = p_{21}$ property as follows,

$$p_{11} = p_1 V_1 + p_2 V_2 + \langle p_{13} N_{,1} \rangle, \tag{5}$$
$$p_{12} = p_{21} = \langle p_{13} N_{,2} \rangle = \langle p_{23} N_{,1} \rangle, \tag{6}$$
$$p_{22} = p_1 V_1 + p_2 V_2 + \langle p_{23} N_{,2} \rangle, \tag{7}$$

where ${}_{13}N$ and ${}_{23}N$ are functions of complex variable $z = y_1 + iy_2$ that are solutions of the local problems ${}_{13}L$ and ${}_{23}L$, respectively, V_1 and V_2 are the percentages of concentrations or matrix and fiber in the composite, $V_1 + V_2 = 1$. The angular brackets define the volume average per unit length over the area V of the cell, that is $\langle F \rangle = \frac{1}{V} \int_V F(y) dy$.

3. Antiplane problems ${}_{\alpha 3}L$ ($\alpha = 1, 2$)

A plane section of a unidirectional fiber composite with crossing angle θ embedded in a matrix is shown in Fig. 1. The two-phase periodic composite considered here consists of parallelogram arrays of identical parallel circular cylinders embedded in an elastic homogeneous medium (Fig. 2). As a unidirectional fibrous composite is assumed, the microstructure of the composite along the third direction (perpendicular to plane of cross-section) remains constant. The fibers are straight of circular cross sections with radius R_0.

The mathematical statement of the problem consists in finding doubly periodic harmonic functions $N(y_1, y_2)$, (the subscripts before the local functions will be omitted), $N = N^{(\gamma)}(y_1, y_2)$, $(y_1, y_2) \in S_\gamma$ ($\gamma = 1, 2$), that satisfies the Laplace equations

$$\nabla^2 N^{(\gamma)} = 0 \quad \text{in } S_\gamma \text{ regions}, \tag{8}$$

where ∇^2 is the Laplace's operator and the local function N has null average over the periodic cell S, i.e. $\langle N \rangle = 0$ and satisfies the following perfect conditions at the interface Γ

$$\|N\| = 0 \quad \text{on } \Gamma, \tag{9}$$
$$\|p(N_{,1} n_1 + N_{,2} n_2)\| = -\|p\| n_\alpha \quad \text{on} \quad \Gamma, \tag{10}$$

now, the comma notation denotes a partial derivative relative to the y_γ component. The outward unit normal vector to the interface Γ is $\mathbf{n} = (n_1, n_2)$.

The problem (8)–(10) should be converted into dimensionless problems (${}_{\alpha 3}L$) making the appropriate change. Defining the dimensionless variable $\boldsymbol{\xi} = \mathbf{y}/l$, then $N^{(\gamma)}_{,i} = u^{(\gamma)}_{,i}$ and $N^{(\gamma)}_{,ii} = u^{(\gamma)}_{,ii}/l$ where $u^{(\gamma)} = N^{(\gamma)}/l$ and the $u_{,i}$ is the derivative respect to the variable ξ_i.

The dimensionless problems over de periodic cell $S = S_1 \cup S_2$ are now written below

$$\nabla^2 u^{(\gamma)} = 0, \quad \text{in } S_\gamma, \tag{11}$$

$$\|u\| = 0, \quad \text{on } \Gamma, \tag{12}$$

$$\left\| \frac{p(u_{,1}n_1 + u_{,2}n_2)}{p_1} \right\| = -(1-\rho)n_\alpha, \quad \text{on } \Gamma, \tag{13}$$

where $\rho = p_2/p_1$.

4. Solution of the local problems $_{\alpha 3}L$ ($\alpha = 1, 2$)

The well-developed theory of analytical functions in [34] can be applied to solve this problem. Doubly periodic harmonic functions are to be found for the $_{\alpha 3}L$ local problems in terms of the following expansions of harmonic functions $u^{(\gamma)} = \mathrm{Re}\{\varphi_\gamma(z)\}$, where the symbols Re or Im denoted the real or imaginary part of the complex number, respectively, $\varphi_1 = \frac{a_0 z}{R} + \sum_{k=1}^{\infty}{}^{o} \left(\frac{\zeta^{(k-1)}(z/R)}{(k-1)!}\right) a_k$ and $\varphi_2(z) = \sum_{p=1}^{\infty}{}^{o} \left(\frac{z}{R}\right)^p b_p$. $R = R_0/l$, R_0 is the true radius of the fibers in the composite, and ζ is the Zeta quasi periodic Weierstrass function defined as $\zeta(z) = \frac{1}{z} + \sum_{m,n}^{\infty}{}' \left(\frac{1}{z-P_{nm}} + \frac{1}{P_{nm}} + \frac{z}{P_{nm}^2}\right)$, $P_{nm} = nw_1 + mw_2$, for $m, n \in Z$ and the prime over the summation symbol means that the pair $(m, n) = (0, 0)$ is excluded. The Laurent's expansions of function $\varphi_1(z)$ is given by the following expression

$$\varphi_1(z) = \frac{z}{R} a_0 + \sum_{p=3}^{\infty}{}^{o} \left(\frac{R}{z}\right)^p a_p - \sum_{p=1}^{\infty}{}^{o} \sum_{k=1}^{\infty}{}^{o} \left(\frac{z}{R}\right)^p \sqrt{\frac{k}{p}} w_{kp} a_k, \tag{14}$$

$w_{kp} = \frac{(k+p-1)!}{(k-1)!(p-1)!} \frac{R^{k+p}}{\sqrt{kp}} S_{k+p}$, $S_{k+p} = \sum_{m,n} (mw_1 + nw_2)^{-(k+p)}$, $m^2 + n^2 \neq 0$, $k + p > 2$, by definition $S_2 = 0$. The constants a_o, a_p, b_p and z are complex numbers; the over bar indicate complex conjugate and the superscript "o" on the summation indicates that the sum is carried out only over odd indices, w_1, w_2 are the periods.

The equalities (12) and (13) can be written in terms of the complex functions φ_γ

$$[\varphi_1(z) + \bar{\varphi}_1(z)]_\Gamma = [\varphi_2(z) + \bar{\varphi}_2(z)]_\Gamma, \tag{15}$$

$$[\varphi_1(z) - \bar{\varphi}_1(z) + (1-\rho)((z-\bar{z})\delta_{1\alpha} - i(z+\bar{z})\delta_{2\alpha})]_\Gamma = \rho[\varphi_2(z) - \bar{\varphi}_2(z)]_\Gamma. \tag{16}$$

From the conditions (14) and (15), one obtains

$$b_p = \left(a_0 \delta_{1p} + \bar{a}_p - \sum_{k=1}^{\infty}{}^{o} \sqrt{\frac{k}{p}} w_{kp} a_k \right). \tag{17}$$

From (14) and (16), it is also obtained

$$\rho b_p = \delta_{1p} a_0 - \bar{a}_p - \sum_{k=1}^{\infty}{}^{o} \sqrt{\frac{k}{p}} w_{kp} a_k + R(1-\rho)\delta_{1p}(\delta_{1\alpha} - i\delta_{2\alpha}). \tag{18}$$

Combining (17) and (18) the following infinite system of equations for the unknowns a_k is obtained

$$\bar{a}_p - \chi a_0 \delta_{1p} + \chi \sum_{K=1}^{\infty}{}^{o} \sqrt{\frac{k}{p}} w_{kp} a_k = \chi R \delta_{1p}(\delta_{1\alpha} - i\delta_{2\alpha}), \tag{19}$$

with $\chi = \frac{1-\rho}{1+\rho}$.

From the condition of double periodicity of the dimensionless function u the following relation is obtained

$$u(z + \omega_\gamma) - u(z) = \mathrm{Re}\left(\frac{\omega_\gamma}{R} a_0 + R\delta_\gamma a_1\right) = 0 \quad \text{with } \delta_\gamma = \zeta(z + w_\gamma) - \zeta(z), \tag{20}$$

and solving the above system of algebraic equations, we have that $a_o = -\chi R^2 H_1 \bar{a}_1 - \chi R^2 H_2 a_1$, where $H_1 = \frac{\bar{\delta}_1 \bar{w}_2 - \bar{\delta}_2 \bar{w}_1}{w_1 \bar{w}_2 - w_2 \bar{w}_1}$, $H_2 = \frac{\delta_1 \bar{w}_2 - \delta_2 \bar{w}_1}{w_1 \bar{w}_2 - w_2 \bar{w}_1}$. It is convenient to introduce in (19) the set of variables

$$a_k = \frac{c_k}{\sqrt{k}}, \tag{21}$$

and after some algebraic manipulations, the above system (19) can be rewritten in the following form

$$\bar{c}_p + \chi R^2 H_1 \bar{c}_1 \delta_{1p} + \chi R^2 H_2 c_1 \delta_{1p} + \chi \sum_{k=1}^{\infty}{}^{o} w_{kp} c_k = \chi R \delta_{1p}(\delta_{1\alpha} - i\delta_{2\alpha}). \tag{22}$$

In order to find the solution of system (22), it is reduced into two subsystems with separated real and imaginary parts. Denoting $c_k = x_k + iy_k$, $H_\alpha = h_{1\alpha} + ih_{2\alpha}$, $w_{kp} = w_{1kp} + iw_{2kp}$ where x_k, y_k, $h_{1\alpha}$, $h_{2\alpha}$, w_{1kp} and w_{2kp} are real numbers, that represent

the real or imaginary part of complex numbers c_k, H_1, H_2, w_{kp}, respectively and $i = \sqrt{-1}$. Thus, the system (22) is written in matrix form as follows

$$(I + \delta_{1p}\chi R^2 J + \chi W)X = \chi R\delta_{1p}B, \tag{23}$$

where I is the unit matrix, and

$$J = \begin{pmatrix} h_{11}+h_{12} & h_{21}-h_{22} \\ -(h_{21}+h_{22}) & h_{11}-h_{12} \end{pmatrix}. \tag{24}$$

The matrix W is composed of square blocks of order 2, defined by $w_{kp} = \begin{pmatrix} w_{1kp} & -w_{2kp} \\ -w_{2kp} & -w_{1kp} \end{pmatrix}$ with $k = 2t - 1$, $p = 2s - 1$, t, $s = 1, 2, 3\ldots$, the transpose of infinite vectors X and B have the form $X^T = (x_1, y_1, x_3, y_3, \ldots, x_k, y_k, \ldots)$ and $B^T = (\delta_{1\alpha}, \delta_{2\alpha}, 0, 0, \ldots)$.

In order to determine the effective properties, it is necessary to truncate the system of Eq. (23) into an appropriate order $k = N_0$. A very important first approximation is obtained if we consider $N_0 = 1$. In this case, the system (23) is very easy to solve and its solution is

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \chi R(I + \chi R^2 J)^{-1}\begin{pmatrix} \delta_{1\alpha} \\ \delta_{2\alpha} \end{pmatrix}. \tag{25}$$

In general, the system (23) can be solved for x_1 and y_1 in closed form and the essential constant $a_1 \equiv c_1 = x_1 + iy_1$ of the effective coefficients (5)–(7), is obtained

$$c_1 = \chi R(1, i)(I + \chi R^2 J - \chi^2 N_1 (I + \chi W)^{-1} N_2)^{-1} B. \tag{26}$$

In Eq. (26), $(1, i)$ denoted de 1×2-matrix row with components 1 and imaginary unit i, B is a 2×1-matrix column with $B^T = (\delta_{1\alpha}, \delta_{2\alpha})$, $N_1 = W(w_{k1})$ and $N_2 = W(w_{1p})$, $k = 2t + 1$, $p = 2s + 1$, $t,s = 1, 2, 3, \ldots$

5. Effective properties

The Eqs. (5)–(7) are easily transformed applying Green's theorem to the area integrals. Replacing $N^{(\gamma)} = u^{(\gamma)}/l$, $dy_i = l d\xi_i$ and considering the perfect contact condition (12), the effective coefficients p_{11}, p_{12}, p_{21} and p_{22} are connected by the following relations

$$p_{11} - ip_{12} = p_1 V_1 + p_2 V_2 - \frac{\|p_\gamma\|}{V}\left(\int_\Gamma u^{(1)} d\xi_2 + i\int_\Gamma u^{(1)} d\xi_1\right), \tag{27}$$

$$p_{21} - ip_{22} = -i(p_1 V_1 + p_2 V_2) - \frac{\|p_\gamma\|}{V}\left(\int_\Gamma u^{(1)} d\xi_2 + i\int_\Gamma u^{(1)} d\xi_1\right), \tag{28}$$

where V_1 and $V_2 = \pi R^2/V$ are the volume fraction of matrix and inclusion, respectively; and $V = |w_1||w_2|\sin\theta$ denotes the volume of periodic cell. In (27) and (28), $u^{(1)}$ is the solution of the local problem ${}_{13}L$ and ${}_{23}L$, respectively. Due to the orthogonality of the system of functions $\{e^{inx}\}_{n=-\infty}^{\infty}$ in $[0, 2\pi]$ and the Laurent's expansions of function $\varphi_1(z)$ given in (14), the line integrals in 27 and 28 are obtained

$$\int_\Gamma u^{(1)} d\xi_1 = \frac{\pi R}{2i}\left(\bar{a}_1 + a_0 - \sum_{k=1}^{\infty}{}^{*}\sqrt{k} w_{k1} a_k - a_1 - \bar{a}_0 + \sum_{k=1}^{\infty}{}^{o}\sqrt{k}\bar{w}_{k1}\bar{a}_k\right), \tag{29}$$

$$\int_\Gamma u^{(1)} d\xi_2 = \frac{\pi R}{2}\left(\bar{a}_1 + a_0 - \sum_{k=1}^{\infty}{}^{o}\sqrt{k} w_{k1} a_k + a_1 + \bar{a}_o - \sum_{k=1}^{\infty}{}^{o}\sqrt{k}\bar{w}_{k1}\bar{a}_k\right). \tag{30}$$

The following two equations are obtained setting in (19) $p = 1$, $\alpha = 1$ and $\alpha = 2$, respectively, according to the problem ${}_{\alpha 3}L$ to be solved. They are necessary in order to obtain analytical expressions for the effective coefficients,

$$a_0 - \sum_{k=1}^{\infty}{}^{o}\sqrt{k} w_{k1} a_k = \frac{p_1 + p_2}{p_1 - p_2}\bar{a}_1 - R, \tag{31}$$

$$a_0 - \sum_{k=1}^{\infty}{}^{o}\sqrt{k} w_{k1} a_k = \frac{p_1 + p_2}{p_1 - p_2}\bar{a}_1 + iR. \tag{32}$$

Substituting (29) and (30) into (27) and (28) and using (31) and (32), simple analytical formulae for effective properties are deduced depending only on the unknown a_1 as follows

$$p_{11} + ip_{12} = p_1(1 - 2V_2 a_1), \tag{33}$$

where a_1 is given by (26) for $\alpha = 1$,

$$p_{21} + ip_{22} = p_1(i - 2V_2 a_1), \tag{34}$$

where a_1 corresponds to $\alpha = 2$. Substituting (26) into (33) and (34) it yields

$$p_{11} + ip_{12} = p_1\left(1 - 2V_2\chi(1,i)(I + \chi R^2 J - \chi^2 N_1 (I + \chi W)^{-1} N_2)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \tag{35}$$

$$p_{21} + ip_{22} = p_1\left(i - 2V_2\chi(1,i)(I + \chi R^2 J - \chi^2 N_1 (I + \chi W)^{-1} N_2)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right). \tag{36}$$

6. Analysis of the results

(1) It follows from (35) and (36), that the effective coefficients $p_{12} = p_{21}$. Therefore, the overall performance of the composite is generally orthotropic except for hexagonal and square periodic cell where $p_{12} = p_{21} = 0$ and $p_{11} = p_{22}$, which is a transversely isotropic composite. Clearly, if $V_2 = 0$ or $p_1 = p_2$, the properties of a monolithic material $p_{11} = p_{22} = p_1$ are obtained.

(2) The formulae (35) and (36) are simple and easy to implement, but using the first approximation (25), it is possible to obtain simpler expressions for the effective coefficients

$$p_{11} + ip_{12} = p_1\left(1 - 2V_2\chi(1,i)(I + \chi R^2 J)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \tag{37}$$

$$p_{21} + ip_{22} = p_1\left(i - 2V_2\chi(1,i)(I + \chi R^2 J)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right), \tag{38}$$

and good numerical results for low concentrations of fibers are obtained. For composites with hexagonal and square periodic cell, from (37), (38) it follows that $p_{11} = p_{22} = p_1 \frac{p_1(1-V_2)+p_2(1+V_2)}{p_1(1+V_2)+p_2(1-V_2)}$ which is the Mori–Tanaka result as given in [35]. Therefore the often used formula is shown to be a particular case of the present model as a first approximation. In some cases it is very close to the actual value of the property.

(3) The special cases in which the period-parallelogram are: (i) rectangular with the primitive period $w_1 = 1$, $w_2 = ci$, $c \geqslant 1$, (ii) a parallelogram with the primitive period $w_1 = 1$, $w_2 = 1/2+ci$, $c > 0$ or (iii) a rhombic cells with the primitive period $w_1 = e^{i\theta/2}$, $w_2 = e^{-i\theta/2}$. It appears that these three cases are the only ones in which the lattices S_{k+p} of w_{kp} in (14) are real numbers (see [36]). This condition leads to obtain the following analytical expressions for the effective coefficients, $p_{12} = p_{21} = 0$ and

$$p_{11} = p_1(1 - 2V_2\chi/P^+), \quad p_{22} = p_1(1 - 2V_2\chi/P^-), \tag{39}$$

where $P^+ = 1 + \chi\Lambda_1 - \chi^2 N M_+^{-1} N^T$, $P^- = 1 + \chi\Lambda_2 - \chi^2 N M_-^{-1} N^T$, $M_\pm^{-1}$ denotes the inverse of the matrix $M_\pm = (I \pm \chi W)$, I is the unit matrix, the matrix W and the vector N are defined as $W = (w_{1kp})$, $N = (w_{1k1})$ and $\Lambda_1 = R^2\delta_1$, $\Lambda_2 = 2V_2 - R^2\delta_1$ for the cases (i) and (ii) whereas $\Lambda_1 = R^2\mathrm{Im}(\delta_1 w_1)/V - V_2$, $\Lambda_2 = -R^2\mathrm{Im}(\delta_1 w_1)/V - V_2$ for the case (iii).

The analytical expressions studied in Refs. [30–31] for a transversely isotropic composite with square and hexagonal periodic cells are now obtained as different particular cases of (39). For instance, the properties for a composite with square array are obtained when $c = 1$ for case (i), then $\theta = 90°$ or when $c = 0.5$ for case (ii), $\theta = 45°$. Besides, the effective properties for a composite with hexagonal array are obtained for the case (ii) when either $c = \sqrt{3}/6$ or $c = \sqrt{3}/2$ with $\theta = 30°$ or $\theta = 60°$, respectively.

The Keller's reciprocal relations [37] with a simple modification can be easily demonstrated from (39) for the above mentioned cases (i) and (ii). The product $p_{11}(p_1, p_2, V_2) \cdot p_{22}(p_2, p_1, V_2) = p_1 p_2$ is satisfied, where $p_{11}(p_1, p_2, V_2)$ and $p_{22}(p_2, p_1, V_2)$ mean that the effective properties for the material constituents p_{11} and p_{22} are calculated with the same volume fraction and exchanging the roles of the constituents.

(4) Analytical expressions for two important limit cases (a) porous composite with aligned empty fibers and (b) composite with rigid fibers can be studied with formulae (35), (36) taking $\chi = 1$ and $\chi = -1$ for the cases (a) and (b), respectively.

The overall elastic properties are of great interest in mechanics of heterogeneous materials. Some numerical illustrative examples are shown to demonstrate the effectiveness of the present method for the analysis of the behaviour of heterogeneous materials. The principal cell is a parallelogram with fundamental periods w_1, w_2 and a vertex angle θ. In order to determine the effective properties, the system of Eq. (23) is truncated in an appropriate order N_0, as w_{kp} depends on powers of radius R of the fiber; this ensures a rapid convergence to the exact solution. Only higher order system N_0 is necessary for high volume fraction of fibers when the radius is large. Some calculations were made for $N_0 = 30$ because they are performed in modern computers. They provide good accuracy in the same way as in [38].

(5) As a verification of this model, comparisons with the results reported by Jiang and Cheung [7], for two different rhombic cells ($\theta = 45^o$ and $\theta = 75^o$) are provided in Tables 1 and 2. The effective shear elastic moduli p_{11}/p_1 and p_{22}/p_1 are calculated using (39) taking into consideration the periods $w_1 = e^{i\theta/2}$, $w_2 = e^{-i\theta/2}$ and, in this case, the principal axes lie along the diagonal of the rhombic cell. The ratio $p_2/p_1 = 120$ is used for the computation. As can be seen in Tables 1 and 2, the same values of Table 4 pp. 235 by Jiang and Cheung [7] are obtained using the present model and the transformation equation for elastic moduli (see pp. 45, Lekhnitskii [39]) with respect to arbitrary reference axes.

(6) Table 3 shows the results of the doubly periodic voids. Letting in (37), (38) the matrix modulus be $p_1 = 30$ GPa and $\chi = 1$, the present method provides the result for solids with doubly periodic tunnel voids. The variations of the effective

Table 1
Comparison of present model AHM with the reported results by Jiang et al. [29] for periodic cell of 45°.

Volume fraction	p_{22}/p_1		p_{12}/p_1		p_{11}/p_1	
	Jiang	AHM	Jiang	AHM	Jiang	AHM
Angle = 45°						
0.1	1.22306	1.22306	−0.00471	−0.00471	1.21365	1.21365
0.2	1.51608	1.51608	−0.02383	−0.02383	1.46841	1.46841
0.3	1.92172	1.92172	−0.07065	−0.07065	1.78042	1.78042
0.4	2.53343	2.53343	−0.17671	−0.17671	2.18002	2.18002
0.5	3.62069	3.62069	−0.44144	−0.44144	2.73782	2.73782

Table 2
Comparison of present model AHM with the reported results by Jiang et al. [29] for periodic cell of 75°.

Volume fraction	p_{22}/p_1		p_{12}/p_1		p_{11}/p_1	
	Jiang	AHM	Jiang	AHM	Jiang	AHM
Angle = 75°						
0.1	1.21780	1.21780	0.00133	0.00133	1.21852	1.21852
0.2	1.48816	1.48816	0.00672	0.00672	1.49176	1.49176
0.3	1.83374	1.83374	0.01968	0.01968	1.84428	1.84428
0.4	2.29477	2.29477	0.04739	0.04739	2.32016	2.32016
0.5	2.95203	2.95203	0.10602	0.10602	3.00884	3.00884
0.6	4.00126	4.00123	0.23812	0.23812	4.12884	4.12884

Table 3
Effective properties for doubly periodic voids fiber composite for $\rho = 30$.

V_2	p_{22}				p_{11}		p_{12}	
	30°	45°	60°	90°	30°	45°	30°	45°
0.1	24.9171	24.6399	24.5455	24.5453	24.0903	24.4438	−0.23869	−0.09806
0.2	21.1380	20.3053	20.0000	19.9959	18.2941	19.6447	−0.82097	−0.33032
0.3	18.1197	16.7111	16.1533	16.1274	12.3815	15.4377	−1.65648	−0.63666
0.4	15.4583	13.6541	12.8534	12.7605	5.0993	11.6744	−2.99039	−0.98925

longitudinal shear moduli p_{11}, p_{12}, p_{22} for four different periodic cells are shown. For hexagonal and square arrays, similar values to those reported in Table 7 by Jiang and Cheung [7] are obtained for void volume fraction. A decrease in the material stiffness by square or hexagonal array of tunnel voids resulted with transversely isotropic symmetry. The effective longitudinal p_{22} modulus of a solid with voids decreases as the angle of the periodic cell increases. The effective shear modulus of the array with the periodic cell with $\theta = 30°$ is higher than voids in a square array when the void volume fraction is the same. The opposite is true for a soft matrix and hard inclusions. For the effective p_{11} modulus the situation is reversed, this ratio increases proportionally to the angle of the periodic cell. p_{12} is negative for $0 < \theta < 60°$ and positive for $60° < \theta < 90°$.

(7) Analytical expressions (39) can be used to validate experimental results and theoretical models. In the case of perfectly rigid inclusions ($\rho = \infty$), the present model is compared in Fig. 3 with the experimental and analytical results reported in Ref. [38], numerical results of Refs. [4,40]. The models accurately predict the effective properties with a negligible overestimation, and notable discrepancy only appears for the largest percentages of the fiber concentration. Note that the simple formulae (37) and (38) demonstrated its validity even for volume fraction of 0.7. The expressions (35) and (36) are used for getting a good estimate of the effective properties for volume fraction values close to percolation limit.

(8) The effective properties behaviour when $0 \leqslant \rho \leqslant \infty$ is studied in Fig. 4 for various orthotropic composites. This figure is inspired on Fig. 5 of [4]. In this case, a small increase in angle in the periodicity of the fibers greatly increases the coefficients p_{22}/p_1 and p_{12}/p_1. This happens because of the smaller fibers spacing along the lower diagonal of the rhombic cell and $V_2 = 0.49$ is a high volume fraction for the cell with $\theta = 35°$. In the region where $0 < \rho < 1$, the coefficient p_{11}/p_1 exhibits major differences with the case of composites with periodic cell of $\theta = 60°$ and $\theta = 75°$. If $\rho = \infty$ and $V_2 \to V_2^{\max}$ for a specific composite, the present model illustrates that, for very rigid fiber, the composite properties are very sensitive to small changes at high volume fraction, as it is reported in Ref. [4].

(9) In Fig. 5, the effective properties of composite are calculated using (39) for the case when the primitive period-parallelogram is a rectangle, $w_1 = 1$, $w_2 = ci$, $c > 1$ and $\rho = 120$. Note that each cell, characterized by θ, has a different percolation volume, that is, the volume when the fibers are in contact. Thus the plot for each cell has a different ending point, its corresponding percolation volume. The periodic cell is the classical square cell when $c = 1$ and in this composite $p_{11} = p_{22}$. Formula (39) shows that the effective coefficients also depend on the height of the cell c (Fig. 2). Thus, for a fixed fiber volume fraction, the increase in the height c induces the effective property associated with p_{11}/p_1 to increase, resulting in higher

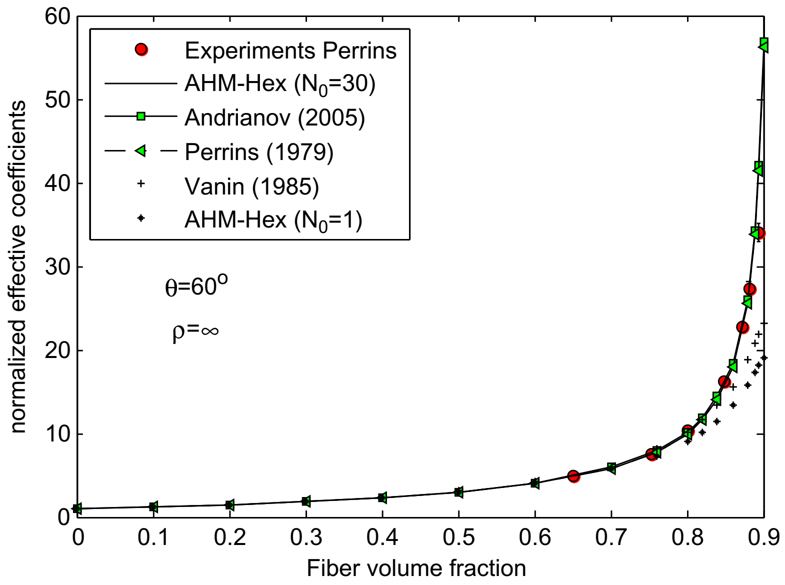

Fig. 3. Comparison of present model AHM with the experimentals and analítical results reported in Perrins et al. [38], numerical results of Vanin [40] and Andrianov [4].

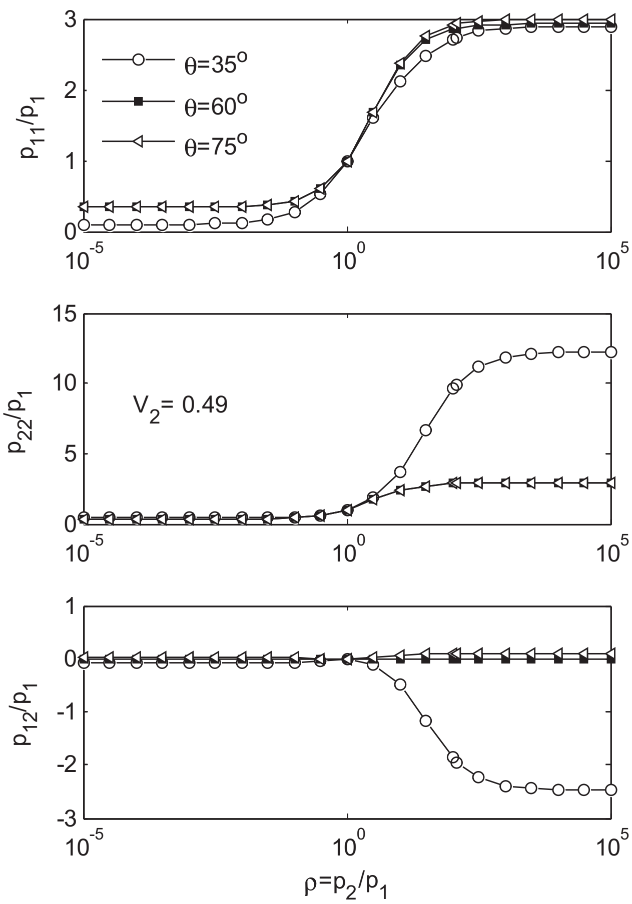

Fig. 4. The behaviour of present solution when $0 \leqslant \rho \leqslant \infty$ for three composites with differents periodicity.

values for the square cell composite denoted in this figure by square-bold. On the other hand, the properties associated with p_{22}/p_1 decrease and they are lower than square cell composite properties.

In the case when the primitive period-parallelogram is given by, $w_1 = 1$, $w_2 = 1/2 + ci$, $c > 0$ (see Fig. 2), a similar situation occurs. The case $c = \sqrt{3}/2 \approx 0.866$ represents the hexagonal cell. In Figs. 5 and 6, the curves related to square and hexagonal

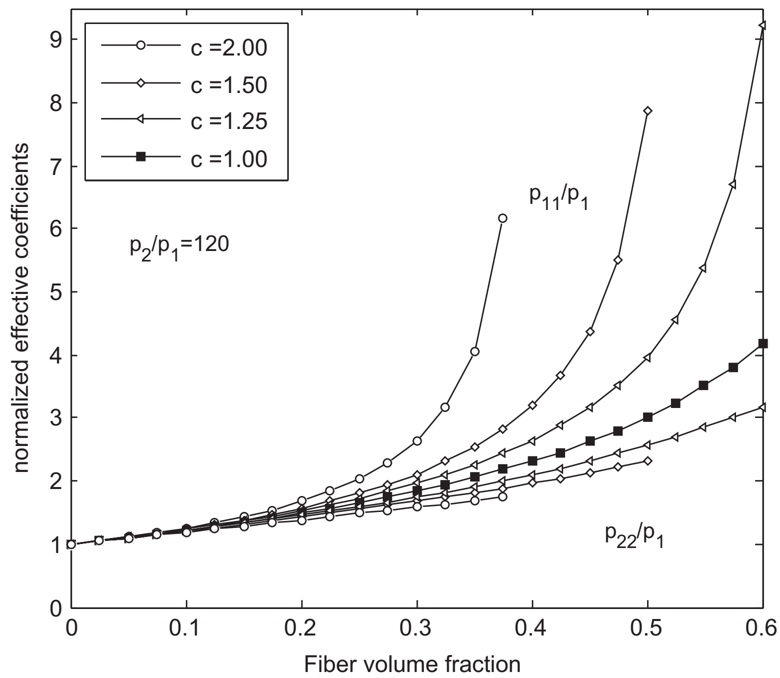

Fig. 5. Effective properties of composite with rectangular period-parallelogram, $w_1 = 1$ and $w_2 = ai$, $a > 1$, calculate for differents values of a and $\rho = 120$.

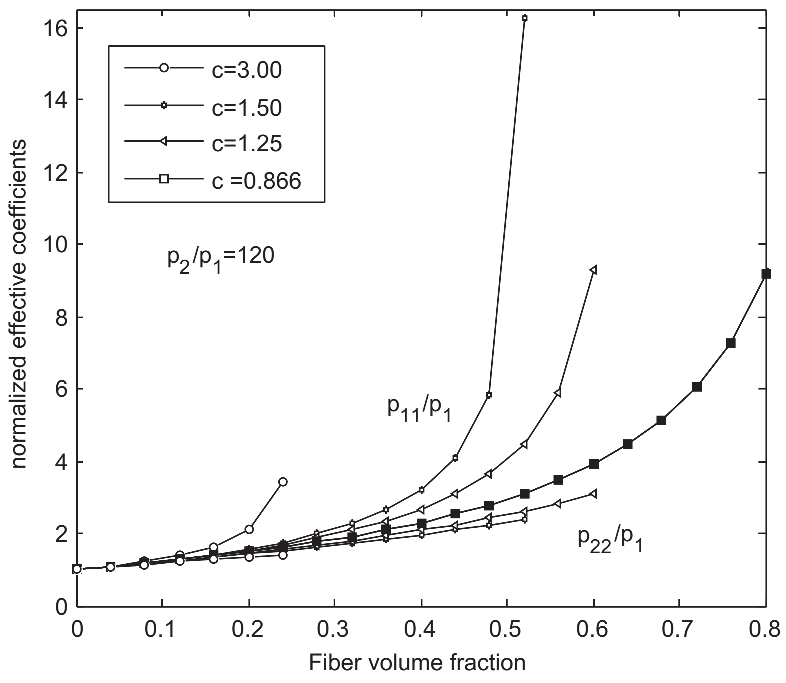

Fig. 6. Effective properties of composite with rhombic period-parallelogram, $w_1 = 1$ and $w_2 = 1/2 + ci$, $c > 0$, calculated for differents values of c and $\rho = 120$.

arrays divide the region occupied by the rectangular figures into two parts (superior and inferior) where the corresponding curves to p_{11} lie in the superior part whereas p_{22} are located in the inferior region for the indicated values of the height c

7. Conclusions

Closed-form analytical formulae of axial effective coefficients for two-phase fibrous elastic composites with isotropic constituents and period-parallelogram cell are given in a unified form. The overall performance of the composite is in general orthotropic; the traditional cases which have been studied (hexagonal and square array) are obtained as particular cases of this more general formulation for the cells. The limit cases of empty and rigid fibers are also considered. The results include comparisons with recent reports in Ref. [29] with an excellent agreement illustrating the efficiency of the current formulae.

The present formula satisfies Keller's reciprocal relations [37]. This connection is true not only for circular interface, but also for other symmetric interfaces. Calculations over all the range of variation of area fiber fraction is possible using the proposed formulae. Comparisons with experimental results reported in the literature revealed a good performance. The formulae are self-contained and are amenable to very simple computations which could be useful for checking numerical and experimental results.

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