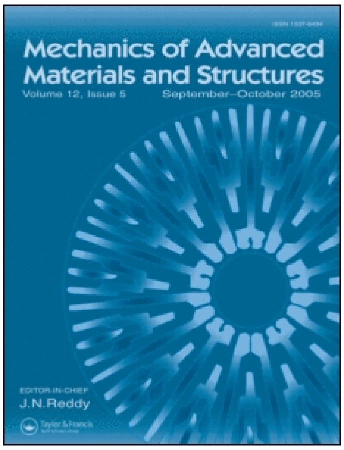

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Effective Properties of Non-Linear Elastic Laminated Composites with Perfect and Imperfect Contact Conditions

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In the present work, the asymptotic homogenization method is applied to investigate the global behavior of non-linear elastic laminated composites exhibiting a periodic structure. The periodic cell can be formed by any finite number of layers. Two different types of non-linear composites are studied: a physically "weakly non-linear" situation given by a third-order polynomial energy, and a "strongly non-linear" situation corresponding to a piece-wise linear constitutive equation (linear hardening). Analytical expressions of the effective laws are derived from the solution of the local problems for three types of different conditions at interface. The influence of two types of imperfect contact conditions at interface is shown by means of numerical examples that involve the measured second-order elastic constant.

Keywords non-linear composite, imperfect contact conditions, effective law

1. INTRODUCTION

Predicting the mechanical behavior of laminated composites is vitally important for efficient design in structural applications. Therefore, it is not surprising that this subject has received much attention since the early days of composite mechanics, even addressed following several different approaches in the context of anisotropic *non-linear* elasticity as they often exhibit *non-linear* behavior in service. *Non-linear* is associated both with the properties of the constituent phases and with the interaction between them. In *non-linear* elasticity, the macroscopic description of the material response is given in terms of a strain-energy function, which is dependent on certain deformation invariants. Recently, different works have been published relating to the *non-linear* behavior of composites [1–9].

Attempts to analyze the effective response of composite materials generally fall within one of the following three categories: i) the calculation of rigorous bounds [10–14]; ii) the simulation of the response by a numerical approximation of the governing equations of an n inclusion-matrix system [15–17]; or iii) the use of ad hoc models which estimate a local field quantity, required for the analysis of the effective property, on the basis of an auxiliary one-inclusion problem [18]. The strengths and weaknesses of these approaches are well known. Thus, although bounds yield rigorous results only based on a limited amount of information (generally independent of microstructural details), large phase stiffness ratios give rise to upper and lower bounds which differ so significantly as to render them useless. Alternatively, numerical simulation based on the exact equations of an n inclusion suspension provides sharp predictions of effective properties up to fairly large volume concentrations; however, such calculations are computationally expensive and hence essentially limited to linear composites. Finally, effective medium models provide simple results for the effective properties; however, their predictions generally differ from one another as a consequence of the particular one-inclusion problem used in the model to estimate the local fields. These approaches are reasonably well developed for linear composites consisting of an elastic matrix embedding randomly—or periodically—distributed elastic inclusions perfectly bonded at the interfaces. Classes of *non-linear composites* have been considered primarily by the first and third approaches and typically involve a hyperelastic matrix response at infinitesimal strain. The single-cell finite element analysis and the micromechanical method of cells often include matrix and interface responses,

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with finite element calculations typically carried out at finite strain.

The macroscopic constitutive relation is derived using asymptotic homogenization theory, which ensures the uniformity of micro-stress and strain tensors in each constituent of a periodic multilayered material [13, 19]. The asymptotic homogenization method (AHM) has advantages and disadvantages (see, for instance [20] and [21]) which can be summarized as follows: a) The AHM is an effective technique for investigating both macroscopic and microscopic properties of periodic structures; b) the method is universal. It is applicable to all kinds of processes that might occur in periodic media, such as elasticity, thermo-elasticity, piezoelectricity, thermo-piezoelectricity, magneto-elasticity, visco-elasticity, plasticity, in bone mechanics and in porous media. See some references and more details in [22]. It is possible to consider both linear and nonlinear models; differential as well as operator equations; c) the AHM is rigorously justified in the works of many authors (see, for instance [20–21]). Theorems have been proven about the solvability of the problems which are elementary steps of an algorithm for constructing an asymptotic solution; d) the main disadvantage of the method is that the numerical solution of cell problems is required. It leads, in general, to the solution of a system of partial differential equations which are complicated; e) another disadvantage is related to the solution of three-dimensional problems on a cell. It takes a lot of computer time and memory capacity.

The formulation of macroscopic non-linear constitutive relations for multilayered materials with perfect contact at the interface has been treated by several authors, for example [23–26]. The application of such models to practice with complex geometry or complex loading conditions is subjected to the hypothesis of perfect bonding at the interface. Indeed, several studies show that the interface properties strongly affect the macroscopic behavior of composite materials, in particular in the case of composite frictional materials such as reinforced soil [27–32].

This work is designed to study the influence of imperfect interfacial contact in the response of laminated composites. It also is an extension of the results reported in [19, 20], where only continuity conditions on the matrix-inclusion contact surface are considered. Here we consider two particular cases of physical nonlinear constitutive relations. This contribution is structured as follows. In Section 2, a summary of the main results of the homogenization method for non-linear elastic composites and their specifications to the layered case are shown. In Sections 3 and 4, one-dimensional weakly and strongly non-linear composites are studied. Analytical expressions of the effective law are derived considering three types of contact conditions (*perfect*, *spring-type* and *membrane-type*). Section 5 provides some concluding remarks.

2. HOMOGENIZATION

The mathematical foundations of the asymptotic homogenization method applied to non-linear heterogeneous media can be found in the classical specialized literature. For instance, in [20, 21], the general formalism of homogenization is shown for an infinite order of accuracy with respect to the small geometrical parameter with a rigorous justification, and a proof of conservation of convexity and symmetry properties for the effective tensor is also derived following a variational approach. For reasons of completeness, a summary of the main results that will be used is outlined here. A specification to a laminated composite following [19] is also included in this section.

2.1. On Homogenization and Effective Laws of Non-Linear Periodic Composites

Let us consider a non-linear heterogeneous anisotropic material occupying a bounded regular domain $V \subset \mathbb{R}^3$ with a smooth boundary $\Sigma = \partial V$. We suppose that the medium exhibits a periodic structure characterized by a small geometrical parameter $\alpha = l/L$ where l and L are typical length scales associated with the periodic cell Ω and the region V, respectively. The boundary-value problem is given by

$$\frac{\partial}{\partial x_j}\sigma_{ij}\left(\vec{\xi}, \nabla_x \vec{u}\right) + X_i = 0, \quad \text{in } V \quad \vec{u} = 0, \quad \text{on } \Sigma \tag{1}$$

where $\vec{x} = (x_1, x_2, x_3)$ and $\vec{\xi} = (\xi_1, \xi_2, \xi_3)$ with $\vec{\xi} = \vec{x}/\alpha$ are the global and local coordinates respectively, and, σ_{ij} is a symmetric tensor-valued stress function defined on the space of deformation gradients, $\nabla_x \vec{u}$ is the 3×3 gradient of the displacement vector $\vec{u} = (u_1, u_2, u_3)$ with elements $\partial u_i/\partial x_j$ and the x_i's are the components of body force vector. Let us suppose that σ_{ij} is an Ω-periodic function of the arguments $\vec{\xi}$ and $\partial u_i/\partial x_j$. The Latin indices run from 1 to 3. The homogenization method (see, for instance [21], for a rigorous mathematical description) allows us to transform Eq. (1), using $u_i(\vec{x}, \vec{\xi}) \sim v_i(x) + \sum_{q=0}^{\infty} \alpha^q \, N_i^{(q)}(\vec{\xi}, \nabla_x \vec{v})$ ($N_i^{(q)}$ denotes the local function of q order) into the following averaged problem

$$\frac{\partial}{\partial x_j}\vec{\sigma}_{ij}(\nabla_x \vec{v}) + X_i = 0, \quad \text{in} \quad V \quad \text{and} \quad v = 0, \text{on } \sigma$$

where $\vec{\sigma}_{ij}$ denotes the effective stress defined by the following average over the periodic cell Ω:

$$\vec{\sigma}_{ij}(\nabla_x \vec{v}) = \langle \sigma_{ij}(\vec{\xi}, \nabla_x \vec{v} + \nabla_\xi N^{(1)}) \rangle \tag{2}$$

where $N^{(1)}(\vec{\xi}, \nabla_x \vec{v})$ is an Ω-periodic solution of the equation of the local problem [22, 32], that is

$$\frac{\partial}{\partial \xi_j}\sigma_{ij}(\vec{\xi}, \nabla_x \vec{v} + \nabla_\xi N^{(1)}) = 0, \quad \text{on } \Omega \tag{3}$$

This system of equations (depending of the parameter $\nabla_x v$) must be complemented with additional appropriate interface

conditions at the surfaces of discontinuity as will be shown in Section 2.3.

2.2. Application to Non-Linear Periodic Laminated Composites

In this subsection Eqs. (2) and (3) will be reduced to the case of non-linear laminated composites following the description given in [19, 33]. Let us consider a non-linear laminated body in which the periodic cell $\Omega = \{\xi = (\xi_1, 0, 0) : 0 \leq \xi_1 \leq 1\}$ is made of a finite number n of layers which are perpendicular to the x_1-axis. In this case, the functions $\sigma_{ij}(\xi, \nabla_x \vec{u})$ and $N_i^{(1)}(\vec{\xi}, \nabla_x \vec{v})$ depend on the fast coordinate $\xi = (\xi_1, 0, 0)$ and consequently Eq. (3) can be expressed as

$$\frac{d}{d\xi}[\sigma_{i1}(\xi v_{p,1} + N_p^{'}, v_{p,2}, v_{p,3})] = 0, \tag{4}$$

where the prime and the comma denote differentiation with respect to $\xi \equiv \xi_1$ and x_i, respectively and $N_i^{(1)} \equiv N_i$. Eq. (4) is equivalent to

$$\sigma_{i1}(\xi; v_{p,1} + N_p^{'}, v_{p,2}, v_{p,3}) = A_i(x), \tag{5}$$

where $A_i(x)$ are the components of a certain function to be determined. Assuming sign definiteness for the Jacobian of σ_{ij}, we obtain from Eq. (5):

$$v_{i,1} + N_i^{'} = \sigma_{i1}^{-1}(\xi; A_p(x), v_{p,2}, v_{p,3}), \tag{6}$$

where the function σ_{i1}^{-1} denotes the inverse function of σ_{i1}. Now, by averaging Eq. (6) and taking into account the periodicity of N_i the function $A_i(x)$ can be obtained from this last expression as follows

$$v_{i,1} = \left\langle \sigma_{i,1}^{-1}(\xi; A_p(x), v_{p,2}, v_{p,3}) \right\rangle, \quad A_i(x) = \langle \sigma_{i1}^{-1} \rangle^{-1}(v_{p,q}).$$

Now, the components of the local function $N(\xi, \nabla_x v)$ can be calculated, on each interval $[0; \xi]$ with $\xi \in \Omega$, by using the following formulae:

$$N_i(\xi, \nabla_x v) = \int_0^{\xi} Q_{i1} d\xi - \left\langle \int_0^{\xi} Q_{i1} d\xi \right\rangle + (0.5 - \xi) v_{i,1}, \tag{7}$$

where

$$Q_{i1} \equiv \sigma_{i1}^{-1}\left(\xi; \left\langle \sigma_{p1}^{-1} \right\rangle^{-1}(v_{q,r}), v_{p,1}, v_{p,2}\right). \tag{8}$$

Finally, the effective law can be calculated from Eq. (3) as follows $\vec{\sigma}_{ij}(\nabla_x v) = \langle \sigma_{ij}(\xi; Q_{p1}, v_{p,2}, v_{p,3}) \rangle$.

Formulae (7) and (8) involve certain functions independent of ξ which play an important role when different contact conditions at the interface are considered.

2.3. Mathematical Formulation of Contact Conditions

In general, laminated composites are often analyzed for their in-plane and out-of-plane components, which yield to fully three-dimensional problems. We consider the classical *perfect contact* condition (Mode I), which implies continuity of tractions and displacements at the interface of the composite,

$$[[\vec{u}]] = 0, \tag{9}$$

$$[[\sigma_{ij}]] n_j = 0, \tag{10}$$

Moreover, the *spring-type* imperfect contact condition [27, 28, 30, 31, 34] is analyzed. At the interface, continuity of tractions is maintained but there exist jumps in the displacement, such that the jumps in the tangential and normal tractions depend on the imperfection parameters M and K respectively. The boundary conditions at the interface are:

$$[[\sigma_{ij}]] n_j = 0, \tag{11}$$

and

$$\sigma_t = M[[u_t]], \quad \sigma_n = K[[u_n]], \tag{12}$$

where the double bracket denotes the jump of the function across the interfaces, that is $[[f]] = f^{(1)} - f^{(2)}$; the parameters M and K are factors of proportionality, which represent the degree of bonding at the interface; and have dimension of stress divided by length. The magnitudes u_n, σ_n and u_t, σ_t denote the displacement and traction vectors respectively, in the normal and tangential directions at the interface. Often it is called Mode II imperfection.

Although the *membrane-type* imperfect contact is not realizable in one-dimensional composites, it is interesting from a mathematical point of view. The *membrane-type* imperfect contact [19, 27, 30] can be formulated in the following form:

$$[[\vec{u}]] = 0 \tag{13}$$

and

$$[[\sigma_{i1} + a u_i]] = 0 \; (i = 1, 2, 3), \quad [[\sigma_{i\gamma}]] = 0 \; (\gamma = 2, 3) \tag{14}$$

where the parameter "a" denotes certain magnitude related to imperfect equilibrium at the interface (Mode III).

Notice that the classical *perfect contact* condition (Eqs. (9) and (10)) can be derived when $a = 0$.

In the case of relatively thin-section composites, the in-plane components are pronounced and two-dimensional analyses are often considered. In this work, for the sake of simplicity, loading a bar by normal stresses at its end is considered. Here, $\sigma_{11} \equiv \sigma$, $u_1 \equiv u$ are the only non-zero components of the stress function and the displacement vector, respectively, and they depend only on $x_1 \equiv x$ ($\sigma_{12} = \sigma_{13} = \sigma_{23} = \sigma_{22} = \sigma_{33} = 0$, $u_2 = u_3 = 0$). Thus, using the asymptotic homogenization

method the condition the expression (11)–(14) for *spring* and *membrane type* can be rewritten in the form

$$\sigma = \mathrm{K}\,[[N]]. \tag{15.1}$$

Whereas, the expression (13)–(14) for *membrane type* can be rewritten in the form

$$[[\sigma + aN]] = 0, \quad [[N]] = 0. \tag{15.2}$$

3. WEAKLY NON-LINEAR COMPOSITES

In the present example we study the behavior of a particular kind of laminated composite, namely, one-dimensional *non-linear* elastic composite, for instance, an elastic bar made of different sectional materials. This means that we are extracting from the laminated composite structure, a thin layer parallel to the x_1-axis (see Figure 1). We are studying an extension of the results reported [19, 20] where only perfect contact conditions were considered. Now, we study in detail the "weakly non-linear" composite, that is, a one-dimensional approximation of a polynomial constitutive relation [35, 36].

The constituents of the composite are considered elastic, homogeneous. The periodic cell has n phases along the x_1-axis (see Figure 2).

The "weakly non-linear" constitutive law can be written in the following form [14, 35, 36]. The phase constitutive law is taken as

$$\sigma^{(\tau)}(\xi, u') = \alpha_\tau u' + \beta_\tau (u')^2 \quad \text{for } \tau = \overline{1,n} \tag{16}$$

where $\alpha_\tau = C_{11}^{(\tau)}$ and $\beta_\tau = \frac{1}{2}(3C_{11}^{(\tau)} + C_{111}^{(\tau)})$ are the material constants of phase τ. Here $C_{11}^{(\tau)}$, $C_{111}^{(\tau)}$ are second- and third-order elastic coefficients, and β_τ is an elastic coefficient usually found in the non-linear acoustics of shock waves [36].

In the following subsections, the effective "weakly non-linear" law for different types of contact conditions are derived analytically.

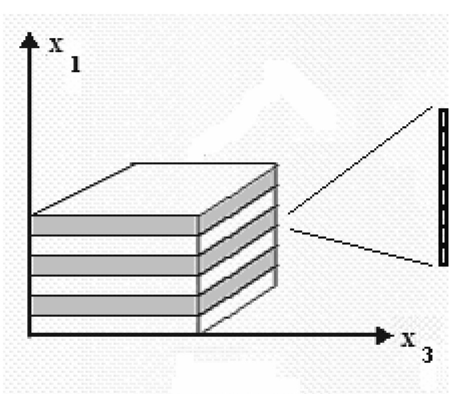

FIG. 1. Laminated composite structure.

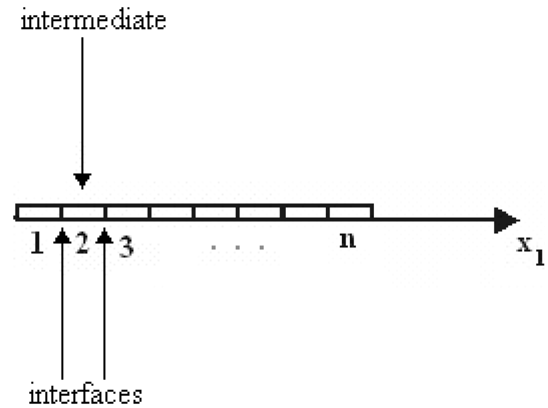

FIG. 2. One-dimensional composite.

3.1. Perfect Contact Conditions (Mode I)

From Eq. (7) we can write, $N_1(\xi, v'(x)) = C_1(x) - \xi C_2(x)$, where $C_1(x)$ and $C_2(x)$ are arbitrary functions to be determined. From Eq. (4), the local function N_1 satisfies the equation $\frac{\partial}{\partial \xi_j}[\sigma(\xi, y + \frac{\partial N_1}{\partial \xi})] = 0$.

A composite made of homogeneous constituents, for which Eq. (16) holds is considered. For simplicity, in further calculation we consider a three-phase composite with constituents denoted by "1," "2," and "3," respectively.

The local function $N_1(\xi, x) = N_1(\xi)$ can be defined as,

$$N_1(\xi) = \begin{cases} C_1^{(1)} - \xi C_2^{(1)} & \text{if } \xi \in (0; \gamma_1) \\ C_1^{(2)} - \xi C_2^{(2)} & \text{if } \xi \in (\gamma_1; \gamma_2) \\ C_1^{(3)} - \xi C_2^{(3)} & \text{if } \xi \in (\gamma_2; 1) \end{cases} \tag{17}$$

The uniqueness condition for the displacement in terms of the local function can be written as

$$N_1^{(1)}(0) = 0, \tag{18}$$

and the periodicity condition for the local function is

$$N_1^{(1)}(0) = N_1^{(3)}(1). \tag{19}$$

From Eq. (18) we have

$$C_1^{(1)} = 0. \tag{20}$$

Using the conditions (19) and (20) we can write,

$$C_1^{(3)} = C_2^{(3)}. \tag{21}$$

From Eqs. (18) and (19) the displacement contrast can be given as

$$\begin{aligned} [[N_1]]_{\gamma_1} &= -\theta_1 C_2^{(1)} - C_1^{(2)} + \theta_1 C_2^{(2)} \\ [[N_1]]_{\gamma_1+\gamma_2} &= C_1^{(2)} - (\theta_1 + \theta_2)\, C_2^{(2)} - \theta_3 C_2^{(3)}, \end{aligned} \tag{22}$$

where $\theta_1,\ \theta_2$ and θ_3 are the volume fraction of the constituents "1," "2," and "3," respectively. Moreover $\theta_1+\theta_2+\theta_3=1$.

Substituting Eqs. (22), (18) and (19) into Eqs. (9) and (10) (perfect contact conditions) one can obtain:

$$-\theta_1 C_2^{(1)} - C_1^{(2)} + \theta_1 C_2^{(2)} = 0, \quad C_1^{(2)} - (\theta_1+\theta_2) C_2^{(2)} - \theta_3 C_2^{(3)} = 0, \tag{23}$$

and

$$\begin{cases} \sigma^{(1)}\left(v' + \dfrac{\partial N_1^{(1)}}{\partial \xi}\right) = \sigma^{(2)}\left(v' + \dfrac{\partial N_1^{(2)}}{\partial \xi}\right) \\ \sigma^{(2)}\left(v' + \dfrac{\partial N_1^{(2)}}{\partial \xi}\right) = \sigma^{(3)}\left(v' + \dfrac{\partial N_1^{(3)}}{\partial \xi}\right) \end{cases} \tag{24}$$

Now, combining Eq. (24) with Eqs. (16) and (17) we have,

$$\begin{aligned} &\alpha_1\big(v' - C_2^{(1)}\big) + \beta_1\big(v' - C_2^{(1)}\big)^2 = \alpha_2\big(v' - C_2^{(2)}\big) \\ &\quad + \beta_2\big(v' - C_2^{(2)}\big)^2 \\ &\alpha_2\big(v' - C_2^{(2)}\big) + \beta_2\big(v' - C_2^{(2)}\big)^2 = \alpha_3\big(v' - C_2^{(3)}\big) \\ &\quad + \beta_3\big(v' - C_2^{(3)}\big)^2 \end{aligned} \tag{25}$$

The non-linear system of algebraic equations, given by Eqs. (23) and (25), can be solved analytically for the unknown functions $C_1^{(2)}, C_2^{(1)}, C_2^{(2)}$ and $C_2^{(3)}$. This fact involves the roots of a fourth-order polynomial equation whose coefficients depend on $v', \alpha_i,\ \beta_i,\ \theta_i$ with $i = 1,\ 2,\ 3$.

The final expression for the effective law can be written as

$$\bar{\sigma} = \frac{1}{3}\sum_{i=1}^{3}\left[\alpha_i\big(v' - C_2^{(i)}\big) + \beta_i\big(v' - C_2^{(i)}\big)^2\right] \tag{26}$$

In the next subsection, we will describe the main items for the computation of the effective law under *spring-type* imperfect interfacial contact (i.e., Eqs. (11) and (12)).

3.2. Spring Type Imperfect Contact (Mode II)

Condition (15.1) can be expressed as $\sigma(v' + \frac{\partial N_1}{\partial \xi}) = \mathrm{K}[[N_1]]$. Hence,

$$\begin{aligned} \sigma^{(1)}\left(v' + \frac{\partial N_1^{(1)}}{\partial \xi}\right) &= \mathrm{K}[[N_1]]_{\gamma_1}, \\ \sigma^{(2)}\left(v' + \frac{\partial N_1^{(2)}}{\partial \xi}\right) &= \mathrm{K}\,[[N_1]]_{\gamma_1}, \\ \sigma^{(2)}\left(v' + \frac{\partial N_1^{(2)}}{\partial \xi}\right) &= \mathrm{K}[[N_1]]_{\gamma_1+\gamma_2}, \\ \sigma^{(3)}\left(v' + \frac{\partial N_1^{(3)}}{\partial \xi}\right) &= \mathrm{K}[[N_1]]_{\gamma_1+\gamma_2}. \end{aligned}$$

Then, from Eqs. (16) and (22) we have

$$\begin{aligned} &\alpha_1\big(v' - C_2^{(1)}\big) + \beta_1\big(v' - C_2^{(1)}\big)^2 \\ &\quad = \mathrm{K}\big[-\theta_1 C_2^{(1)} - C_1^{(2)} + \theta_1 C_2^{(2)}\big], \\ &\alpha_2\big(v' - C_2^{(2)}\big) + \beta_2\big(v' - C_2^{(2)}\big)^2 \\ &\quad = \mathrm{K}\big[-\theta_1 C_2^{(1)} - C_1^{(2)} + \theta_1 C_2^{(2)}\big], \\ &\alpha_2\big(v' - C_2^{(2)}\big) + \beta_2\big(v' - C_2^{(2)}\big)^2 \\ &\quad = \mathrm{K}\big[C_1^{(2)} - (\theta_1+\theta_2) C_2^{(2)} - \theta_3 C_2^{(3)}\big], \end{aligned} \tag{27}$$

and

$$\begin{aligned} &\alpha_3\big(v' - C_2^{(3)}\big) + \beta_3\big(v' - C_2^{(3)}\big)^2 \\ &\quad = \mathrm{K}\big[C_1^{(2)} - (\theta_1+\theta_2) C_2^{(2)} - \theta_3 C_2^{(3)}\big]. \end{aligned}$$

This system of non-linear equations can be solved for the unknown functions $C_1^{(2)}, C_2^{(1)}, C_2^{(2)}$ and $C_2^{(3)}$. Also, they involve the roots of a fourth-order polynomial equation whose coefficients depend on $K, v', \alpha_i,\ \beta_i,\ \theta_i$ for $i = 1,\ 2,\ 3$.

Thus, the final expression for the effective law has the same expression as Eq. (26), but the coefficients $C_2^{(i)}, i = 1,\ 2,\ 3$ are determining from the system of Eq. (27).

3.3. *Membrane Type* Imperfect Contact (Mode III)

Now, the effective law for *weakly non-linear law* is derived from condition (15.2) using Eqs. (16), (20) and (21). Thus,

$$\begin{aligned} &\alpha_1\big(v' - C_2^{(1)}\big) + \beta_1\big(v' - C_2^{(1)}\big)^2 + a_1 C_2^{(1)} \\ &\quad = \alpha_2\big(v' - C_2^{(2)}\big) + \beta_2\big(v' - C_2^{(2)}\big)^2 + a_2 C_2^{(2)}, \\ &\alpha_2\big(v' - C_2^{(2)}\big) + \beta_2\big(v' - C_2^{(2)}\big)^2 + a_2 C_2^{(2)} \\ &\quad = \alpha_3\big(v' - C_2^{(3)}\big) + \beta_3\big(v' - C_2^{(3)}\big)^2 + a_3 C_2^{(3)}. \end{aligned} \tag{28}$$

Due to the continuity of the displacement, the solution of the non-linear system of equations (Eqs. (13) and (28)) leads to $C_1^{(2)}$, $C_2^{(1)}, C_2^{(2)}$ and $C_2^{(3)}$. These expressions are related to the roots of a fourth-order polynomial equation.

Therefore the general expression of the effective *weakly non-linear* law can be given by formula (26) where the coefficients $C_2^{(i)}, i = 1, 2, 3$ are determined from Eq. (28).

3.4. Analysis of the Results

3.4.1. *Limit Cases*

A qualitative analysis of limit cases of the effective law (26) with *spring-type* interfacial conditions was carried out by considering different selections of material constants. For instance, a) $\alpha_1 = \alpha_2 = \alpha_3 = \alpha \neq 0$ and $\beta_1 = \beta_2 = \beta_3 = \beta \neq 0$; b) $\alpha_1 = \alpha_2 = \alpha_3 = \alpha \neq 0$ and $\beta_1 = \beta_2 = \beta_3 = \beta \to 0$; c) $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\beta_1 = \beta_2 = \beta_3 = \beta \neq 0$; d) $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\beta_1 \neq 0, \beta_2 \neq 0,\ \beta_3 \neq 0, \beta_1 \neq \beta_2$, $\beta_1 \neq \beta_3,\ \beta_2 \neq \beta_3$.

It is known that, for infinitesimal deformations, the constitutive relation must be linear. Therefore, the cases c) and d) are irrelevant or without physical meaning. For small deformations, the cases a) and b) reproduce the linear behavior of the constitutive relation. Conversely, for the finite strain the case b) is out of the scope, and case a) is applicable but it does not satisfy all *weakly non-linear* particular requirements. Only one effective constitutive law remains physically valid for any arbitrary selection of material constants. It can be seen from case e) $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\alpha_3 \neq 0$, $\alpha_1 \neq \alpha_2$, $\alpha_2 \neq \alpha_3$, $\alpha_1 \neq \alpha_3$ and $\beta_1 = \beta_2 = \beta_3 = \beta \to 0$ that the effective law (26) reproduces all particular limit cases including the effective linear law: $\bar{\sigma} = (\frac{\theta_1}{\alpha_1} + \frac{\theta_2}{\alpha_2} + \frac{\theta_3}{\alpha_3})^{-1} v'$.

Moreover, from formula (26) (effective law for Mode II), as $K \to \infty$, one can obtain the law corresponding to perfect contact condition (Mode I). It is confirmed in [31].

On the other hand, the effective law of elastic composites for perfect contact condition at the interface also can be derived using the algorithm reported in [20]. In that case, the effective law in agreement with Eq. (26).

3.4.2. *Numerical Example*

In order to show the global responses of the composite, we illustrate its behavior for different combination of materials. The material coefficients are taken from [37–39] (see Table 1). Graphically, the influence of the volume fraction and the linear coefficients α_τ are not significant whereas variation of the non-linear parameter β_τ is important in the response of the composite.

In Figures 3 and 4, the used volume fractions are $\theta_1 = 0.1$, $\theta_2 = 0.3$ and $\theta_3 = 0.6$. The computations were done for input parameter $K = 10^6$. The physical meaning of this K parameter can be interpreted as certain elastic imperfection that exists between the constituents of the composite. For instance, in the most practical cases, the weak interface response leads to the development of dislocations and voids, this dramatically decreases the strength of the whole material.

In Figure 3, the influence of different types of contact (Mode I, II, III) given by Eq. (26) is shown for a CdTe/GaAs/Glass three-phase composite. The computations were carried out with the above mentioned input parameters and $a_1 = 10^6$, $a_2 = .9\,a_1$, $a_3 = .1\,a_1$. In this figure, note the remarkable distinction of imperfect conditions with respect to perfect condition. The figure shows that *spring-type or membrane-type* imperfect contacts make the composite yielding in comparison with the one with

TABLE 1
Material parameters used in the computations

No.	Materials	α	β
1	CdTe	54 GPa	136.5 GPa
2	GaAs	119 GPa	198 GPa
3	Glass	85 GPa	400 GPa

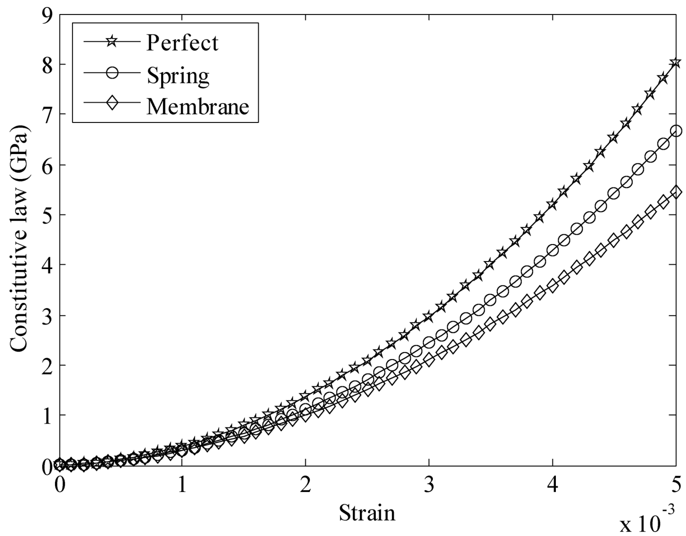

FIG. 3. Comparison between the effective laws for perfect contact and imperfect contact condition at the interface (Mode I, II, III) for a three-phase composite.

perfectly bonded phases. Notice that the curves of the effective laws are parabolic with dilatation coefficients depending on the imperfect parameter. The curve for perfect contact is above the curves for imperfect contact. From numerical experiments we observe that the curves for imperfect contact (Mode II) approach the curve of perfect contact as $K \to \infty$ and the curves for imperfect contact (Mode III) approach the curve of perfect contact as $a_i \to 0 \quad i = 1, 2, 3$.

Although the figure is not shown, a comparison between the three-layer composite (CdTe/GaAs/Glass) and the two-layer composite CdTe/Glass for *imperfect contact* (Mode II, III) shows

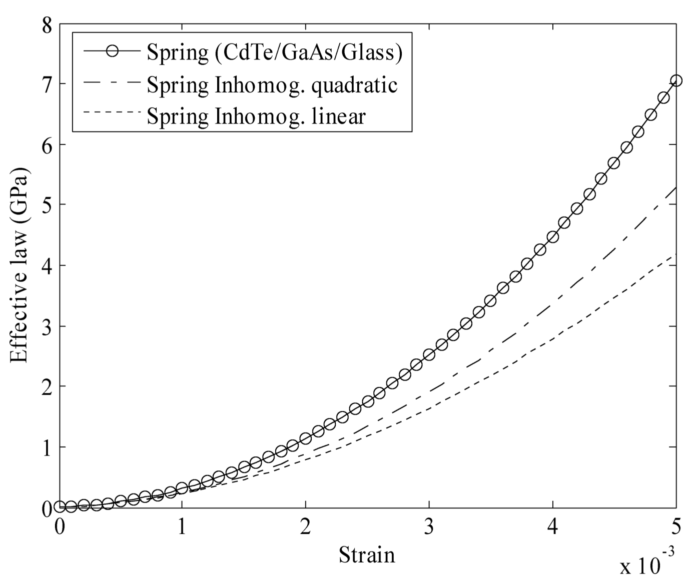

FIG. 4. Comparison between the effective laws for imperfect contact at the interface (Mode II) and inhomogeneous intermediate constituent.

that for the limit case when the intermediate volume fraction in the CdTe/GaAs/Glass composite approaches zero, the three-phase CdTe/GaAs/Glass composite reduces to a two-phase CdTe/Glass composite.

On the other hand, Figure 4 shows the effect of the inhomogeneous intermediate constituent on the obtained effective law for three-layer composites with *spring-type* imperfect contact. In this case, we consider arbitrary variations in the intermediate constituent, analogous to [40]. For example, for mathematical simplicity we choose the quadratic variation of the intermediate constitutive law, i.e., $\sigma = pv' + qv'^2$ where p, q are constants which are determined from the conditions on the elastic constants at the interface of the composite (CdTe/Inhomogenous/Glass). Moreover, linear variation of the intermediate constituent is considered, i.e., $\sigma = mv' + n$ where m, n are calculated analogous to the constants p and q. Figure 4 shows that the inhomogeneous intermediate law provokes a significant change in the global behavior of the composite. The inhomogeneous intermediate produces a softer response in the laminated composite.

Figure 5 displays the obtained effective energy, the upper bound from [41] (denoted as U_B), the lower bound from [42] (denoted by L_B), and the energies of the constituents. The effective energy coincides with the lower bound. The bounds for two-layer composites were computed as follows:

$$U_B = \frac{1}{2}(\alpha_1\theta_1 + \alpha_2\theta_2)(v')^2 + \frac{1}{3}(\beta_1\theta_1 + \beta_2\theta_2)(v')^3,$$

$$L_B = Q(v')^2 - \left(\frac{QSv'}{2\beta_1} - \frac{S^2\alpha_1}{8\beta_1^2} - \frac{S^3\alpha_1}{24\beta_1^3}\right)\theta_1 - \left(\frac{QSv'}{2\beta_1} - \frac{S^2\alpha_2}{8\beta_1^2} - \frac{S^3\alpha_2}{24\beta_1^3}\right)\theta_2,$$

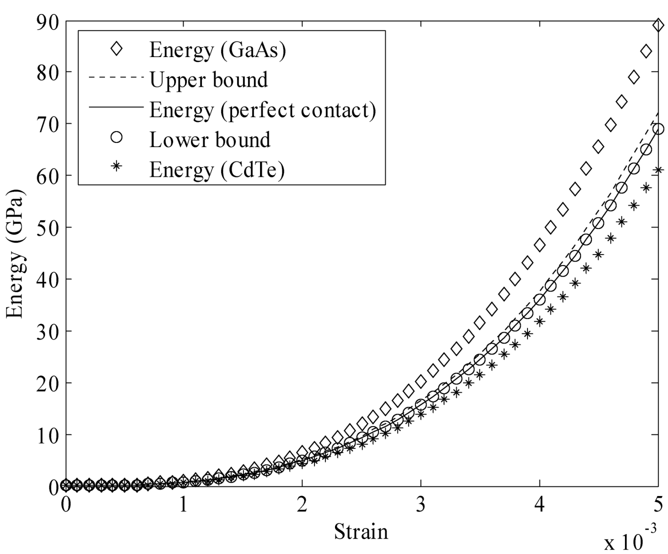

FIG. 5. Lower and upper bounds, effective energy for perfect contact at the interface (Mode I) and the constituent's energies for a two-layer CdTe/GaAs composite.

where $Q = \alpha_2 + \beta_2 v'$ and $S = -\alpha_2 + \sqrt{\alpha_2^2 + 4\beta_2 R v'}$. The used volume fractions are $\theta_1 = 0.6$, $\theta_2 = 0.4$.

In the following section another type of non-linearity is analyzed.

4. STRONGLY NON-LINEAR COMPOSITES

Now, we study in detail a *strongly non-linear* composite. The constituents are considered elastic, homogeneous and responding to the isotropic hardening constitutive laws [43–45]. Also, we study the same (perfect and imperfect) contact conditions as those reported in Section 3. The constitutive laws are derived in the following subsection.

4.1. Constitutive Assumptions

It is well known that, for the ultimate state $M = (\varepsilon_1, \sigma_1)$ reached in simple tension, the elastic limit is σ_1, and the range $\sigma < \sigma_1$ may be termed elastic. If the stress varies within this range, only elastic deformation takes place. Loading beyond state M implies that non-linear deformation will follow.

Under the above assumptions the Hooke's law for the first interval $\varepsilon < \varepsilon_1$, is written in the form $\sigma_1 = E\varepsilon_1$; $E = \frac{\sigma_1}{\varepsilon_1} = \tan A$. On the other hand, for the second interval $\varepsilon \geq \varepsilon_1$ the slope of the straight line is given by $E^* = \frac{\sigma_2 - \sigma_1}{\varepsilon_2 - \varepsilon_1} = \tan B$. In this case, the stress is expressed as $\sigma = E\varepsilon_1 + E^*(\varepsilon - \varepsilon_1)$ (see Figure 6).

In summary, the *strongly non-linear* constitutive relation for the components of a multiphase composite can be written as

$$\sigma^{(\tau)}(u') = \begin{cases} E_\tau u' & \text{if } u' < \varepsilon_s^{(\tau)} \\ E_{\tau^*} u' + (E_\tau - E_{\tau^*})\varepsilon_s^{(\tau)} & \text{if } u' \geq \varepsilon_s^{(\tau)} \end{cases} \tag{29}$$

where E_τ is the Young's modulus for the homogeneous phase τ, E_{τ^*} is the critical elastic Young's modulus for the phase τ

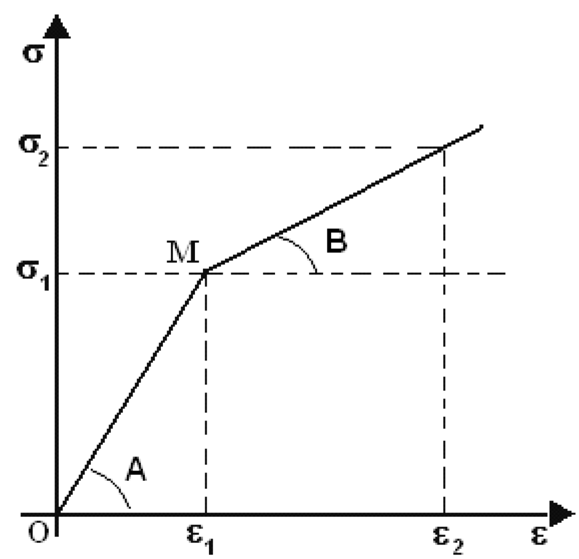

FIG. 6. One dimensional stress-strain hardening.

and $\varepsilon_s^{(\tau)}$ is the strain elastic limit for the phase τ. Eq. (29) corresponds to materials with strain hardening [37], also known as piecewise linear materials or matierals with bi-parabolic potentials.

4.2. Asymptotic Homogenization

Analogous to Section 3, the averaged law for different contact conditions at the interface of the composite is derived. In this sense, the main results for two-layer composites are reported.

Now, we will describe the main steps for the calculation of the effective *strongly non-linear* law considering three types of contact. In this situation, we can distinguish three general behaviors: linear elastic for $v' < \overline{\varepsilon_s^{(1)}}$, linear and *strongly non-linear* elastic for $\overline{\varepsilon_s^{(1)}} \leq v' < \overline{\varepsilon_s^{(2)}}$ and pure *strongly non-linear* elastic composite for $v' \geq \overline{\varepsilon_s^{(2)}}$, where $\overline{\varepsilon_s^{(1)}}$ and $\overline{\varepsilon_s^{(2)}}$ are the strain elastic limits of phases [19, 45].

- Linear elastic behavior $v' < \overline{\varepsilon_s^{(1)}}$.

The perfect contact conditions at the interface (Mode I) are given by Eqs. (9) and (10). Hence, using the asymptotic homogenization method the effective linear law is derived and it can be written as $\bar{\sigma} = \frac{E_1E_2}{E_1\theta_2+E_2\theta_1}v'$. From now on, for the sake of brevity, we introduce the following notation $\langle \bullet, * \rangle = \bullet\theta_1 + *\theta_2$ and thus $\bar{\sigma} = \left\langle E_1^{-1}, E_2^{-1} \right\rangle^{-1} v'$.

The imperfect contact condition (Mode II) is given by Eqs. (11) and (12). Using Eqs. (11) and (29), the formula (12) takes the form:

$$E_1\left(v' - C_2^{(1)}\right) = \mathrm{K}\left[-\theta_1 C_2^{(1)} - \theta_2 C_2^{(2)}\right] \quad \text{and}$$
$$E_2\left(v' - C_2^{(2)}\right) = \mathrm{K}\left[-\theta_1 C_2^{(1)} - \theta_2 C_2^{(2)}\right].$$

The above linear system of equations can be solved and the effective law yields

$$\bar{\sigma} = \left[\left\langle E_1^{-1}, E_2^{-1}\right\rangle + K^{-1}\right]^{-1} v'.$$

Using Eqs. (18)–(20), formula (14) can be expressed for Mode III as

$$\begin{aligned} &E_1\left(v' - C_2^{(1)}\right) - a_1\theta_1 C_2^{(1)} \\ &\quad = E_2\left(v' - C_2^{(2)}\right) - a_2\left(C_1^{(2)} - \theta_2 C_2^{(2)}\right). \end{aligned} \tag{30}$$

The solution of the linear system of Eqs. (13) and (30) provides the effective law in the form,

$$\bar{\sigma} = \frac{1 + \{E_1, E_2\}T_1}{\langle E_1^{-1}, E_2^{-1}\rangle + T_1} v' \quad \text{where } T_1 = \frac{(a_1 - a_2)\theta_1\theta_2}{E_1E_2}.$$

- Linear and *strongly non-linear* elastic $\overline{\varepsilon_s^{(1)}} \leq v' < \overline{\varepsilon_s^{(2)}}$.

From Eq. (10) of Mode I and using Eq. (29) we obtain,

$$\left(v' - C_2^{(1)}\right) + \left(E_1 - E_1^*\right)\varepsilon_s^{(1)} = E_2\left(v' - C_2^{(2)}\right), \tag{31}$$

Replacing Eq. (9) into Eq. (22) we have,

$$-\theta_1 C_2^{(1)} - \theta_2 C_2^{(2)} = 0. \tag{32}$$

The solution of the linear system given by Eqs. (31) and (32) allows us to obtain the effective law in the form $\bar{\sigma} = \langle E_1^{*-1}, E_2^{-1}\rangle^{-1}[v' + (E_1E_1^{*-1} - 1)\theta_1\varepsilon_s^{(1)}]$.

TABLE 2
Constants Δ_i, $(i = 1, \cdots, 5)$ and elastic limit involved in the effective law for three Modes of the contacts

	Mode I	Mode II	Mode III
Δ_1	$\langle E_1^{-1}, E_2^{-1}\rangle^{-1}$	$[\langle E_1^{-1}, E_2^{-1}\rangle + K^{-1}]^{-1}$	$\frac{1+\langle E_1,E_2\rangle T_1}{\langle E_1^{-1},E_2^{-1}\rangle+T_1}$
Δ_2	$\langle E_1^{*-1}, E_2^{-1}\rangle^{-1}$	$[\langle E_1^{*-1}, E_2^{-1}\rangle + K^{-1}]^{-1}$	$\frac{1+\langle E_1^*,E_2\rangle T_2}{\langle E_1^{*-1},E_2^{-1}\rangle+T_2}$
Δ_3	$\frac{(E_1E_1^{*-1}-1)\varepsilon_s^{(1)}\theta_1}{\langle E_1^{*-1},E_2^{-1}\rangle}$	$\frac{(E_1E_1^{*-1}-1)\varepsilon_s^{(1)}\theta_1}{\langle E_1^{*-1},E_2^{-1}\rangle+K^{-1}}$	$\frac{(E_1-E_1^*)(E_2+T_2)\varepsilon_s^{(1)}\theta_1}{\langle E_1^*-1,E_2-1\rangle+T_2}$
Δ_4	$\langle E_1^{*-1}, E_2^{*-1}\rangle^{-1}$	$[\langle E_1^{*-1}, E_2^{*-1}\rangle + K^{-1}]^{-1}$	$\frac{1+\langle E_1^*,E_2^*\rangle T_3}{\langle E_1^*-1,E_2^{*-1}\rangle+T_3}$
Δ_5	$\frac{\left[\begin{smallmatrix}(E_1E_1^{*-1}-1)\varepsilon_s^{(1)}\theta_1\\+(E_2E_2^{*-1}-1)\varepsilon_s^{(2)}\theta_2\end{smallmatrix}\right]}{\langle E_1^{*-1},E_2^{*-1}\rangle}$	$\frac{\left[\begin{smallmatrix}(E_1E_1^{*-1}-1)\varepsilon_s^{(1)}\theta_1\\+(E_2E_2^{*-1}-1)\varepsilon_s^{(2)}\theta_2\end{smallmatrix}\right]}{\langle E_1^{*-1},E_2^{*-1}\rangle+K^{-1}}$	$\frac{\begin{smallmatrix}(E_1-E_1^*)(E_2+T_3)\varepsilon_s^{(1)}\theta_1\\+(E_2-E_2^*)(E_1+T_3)\varepsilon_s^{(2)}\theta_2\end{smallmatrix}}{\langle E_1^{*-1},E_2^{*-1}\rangle+T_3}$
$\overline{\varepsilon_s^{(1)}}$	$\frac{E_1\varepsilon_s^{(1)}}{\langle E_1^{-1},E_2^{-1}\rangle}$	$\frac{E_1\varepsilon_s^{(1)}}{\langle E_1^{-1},E_2^{-1}\rangle+K^{-1}}$	
$\overline{\varepsilon_s^{(2)}}$	$\frac{E_1\varepsilon_s^{(2)}}{\langle E_1^{-1},E_2^{-1}\rangle}$	$\frac{E_1\varepsilon_s^{(2)}}{\langle E_1^{-1},E_2^{-1}\rangle+K^{-1}}$	

where $T_1 = \frac{(a_1-a_2)\theta_1\theta_2}{E_1E_2}$, $T_2 = \frac{(a_1-a_2)\theta_1\theta_2}{E_1^*E_2}$ and $T_3 = \frac{(a_1-a_2)\theta_1\theta_2}{E_1^*E_2^*}$.

Using Eqs. (12) and (29), the formula (22) for Mode II is reduced to

$$E_1^*\big(v' - C_2^{(1)}\big) + \big(E_1 - E_1^*\big)\varepsilon_s^{(1)} = \mathrm{K}\big[-\theta_1 C_2^{(1)} - \theta_2 C_2^{(2)}\big]$$
$$E_2\big(v' - C_2^{(2)}\big) = \mathrm{K}\big[-\theta_1 C_2^{(1)} - \theta_2 C_2^{(2)}\big].$$

Therefore, the solution of this linear system of equations leads to the effective law

$$\bar{\sigma} = \big[\big\langle E_1^{*-1}, E_2^{-1}\big\rangle + K^{-1}\big]^{-1}\big[v' + \big(E_1 E_1^{*-1} - 1\big)\theta_1\varepsilon_s^{(1)}\big].$$

Using Eqs. (18)–(20), the formula (14) for Mode III can be expressed in the form,

$$\bar{\sigma} = \frac{1 + \langle E_1^*, E_2\rangle T_2 v' + (E_1 - E_1^*)(E_2 + T_2)\varepsilon_s^{(1)}\theta_1}{\langle E_1^*{-1}, E_2{-1}\rangle + T_2}$$

where

$$T_2 = \frac{(a_1 - a_2)\theta_1\theta_2}{E_{1^*}E_2}.$$

- Pure *strongly non-linear* elastic composite $v' \geq \overline{\varepsilon_s^{(2)}}$.

In this case, analogous to the above procedure, the effective laws for different modes can be derived.

Finally, the effective law for two-layer composites with different contacts i.e. Modes I, II and III, can be summarized in the following form,

$$\bar{\sigma} = \begin{cases} \Delta_1 v' & \text{if } v' < \overline{\varepsilon_s^{(1)}} \\ \Delta_2 v' + \Delta_3 & \text{if } \overline{\varepsilon_s^{(1)}} \leq v' < \overline{\varepsilon_s^{(2)}} \\ \Delta_4 v' + \Delta_5 & \text{if } v' \geq \overline{\varepsilon_s^{(2)}} \end{cases} \tag{33}$$

The strain elastic limits and the constants Δ_i $(i = 1, \cdots, 5)$ involved in the effective law are listed in Table 2.

4.3. Analysis of Results

The limit cases are analyzed. In Table 2, notice that the coefficients Δ_i corresponding to the *spring-type* interfacial contact are reduced to the coefficients with perfect contact as $K \to \infty$. The continuity of the effective law (33) was proven.

Figure 7 shows the behavior of a two-layer *strongly non-linear* composite made of Al/Steel with perfect (Mode I) and imperfect contact (Mode II) conditions. In Figure 8, before the critical elastic limit of the Steel Young's modulus, there are fourth curves, two curves correspond to the linear elastic behavior for pure constituents, one curve corresponds to the composite with perfect contact and the other curve is for the composite with imperfect contact (Mode II). The second interval (between the critical elastic limit of the Young's modulus for Steel and Al) shows the behavior of the composites and their constituents. The last third interval displays the hardening behavior of the composite after the critical limit of the Young's modulus for both constituents Aluminum and Steel. We must say that these numerical results were compared with finite element calculations carried out by the Chair of Numerical Mechanics of the University of Magdeburg, Germany, as part of a fruitful collaboration developed in the last years [46, 47]. Again the curves show the disagreement between both types of contact. From Table 2, it can be concluded that, in each interval, the slope of the straight line corresponding to the imperfect contact is smaller than the slope of the effective law related to the perfect contact.

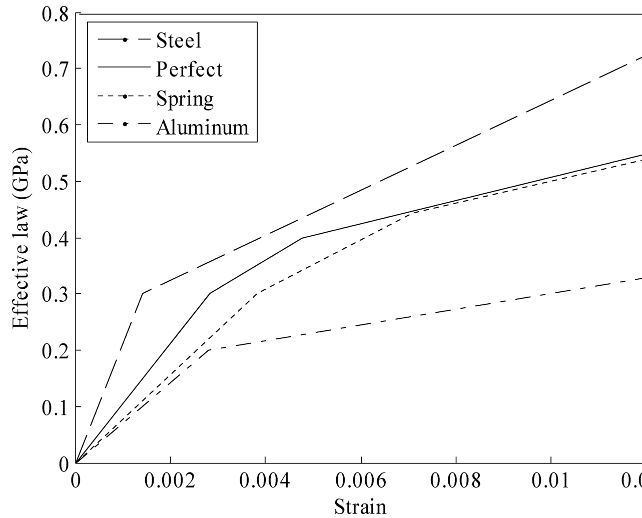

FIG. 7. Comparison between the effective law with perfect contact (Mode I) and imperfect contact (Mode II) for the "strongly non-linear" case.

Therefore, as all curves begin from the same point, the curve of the effective law for imperfect contact is always below the effective law for perfect contact. Moreover, it can be observed that the curves for imperfect contact tend to those for perfect contact as $K \to \infty$. The material coefficients used in the computations were taken from [48] and they are listed in Table 3. The volume fractions used are $\theta_1 = 0.6$, $\theta_2 = 0.4$ and $K = \frac{E_1+E_2}{2}$ where E_1 and E_2 are the Young's modulus of each constituent.

TABLE 3
Material coefficients used in the computations

No.	Material	E	E^*	ε_s	Volume fraction
1	Al	71 GPa	14 GPa	0.0282	0.5
2	Steel	210 GPa	40 GPa	0.00143	0.5

5. CONCLUDING REMARKS

An application of the asymptotic homogenization to *non-linear composites* was presented. The general local problem was deduced and was studied in particular for *non-linear laminate composites*. One-dimensional *non-linear composites* with classical non-linear physical constitutive laws and three types of contact conditions (Mode I, II, III) were investigated. The derived effective non-linear constitutive law discriminates several cases that need special consideration. The imperfect interface (Mode II, III) plays an important role in the global behavior of the composite. Moreover, the inhomogeneity of the intermediate constituents of multi-layer composites may have an influence on their effective law. It should be noted that the contact properties affect the homogenized constitutive relation as shown Figures 3, 4 and 7. The non-linear contribution of the constituents has a significant effect on the properties of the composite under imperfect contact conditions.

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APPENDIX

Symbols

∂V	:	boundary V
$\frac{\partial}{\partial x_j}$	:	partial derivative
u, ξ, x	:	vector
∇_x	:	gradient
$\bar{\sigma}_{ij}$	:	effective stress
$\langle \bullet \rangle$	:	average
N'	:	the ordinary derivative respect to $\xi \equiv \xi_1$
$N_{,i}$	:	partial derivative respect to x_i
$\vec{\sigma}_{il}^{-1}$	:	the inverse function of σ_{il}
$F_{\vert}$	:	partial derivate respect ξ
$E_{(\tau)}$	:	is the Young's modulus for the homogeneous phase τ.
$E^*_{(\tau)}$	:	is the critical elastic Young's modulus for the phase τ.
$\underline{\varepsilon_s^{(\bullet)}}$	:	is the strain elastic limit for the phaseτ.
$\varepsilon_s^{(\bullet)}$	:	is the strain elastic limit for the composite
$\theta_\bullet$	:	volume fraction.
$\alpha_\bullet$	:	linear coefficient
$\beta_\bullet$	:	non linear coefficient
U_B	:	upper bound
L_B	:	lower bound